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A CONTINUOUS TIME APPROACH TO THE PRICING OF BONDS

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This paper develops an arbitrage model of the term structure of interest rates based on the assumptions that the whole term structure at any point in time may be expressed as a function of the yields on the longest and shortest maturity default free instruments and that these two yields follow a Gauss-Wiener process. Arbitrage arguments are used to derive a partial differential equation which must be satisfied by the values of all default free bonds. The joint stochastic process for the two yields is estimated using Canadian data and the model is used to price a sample of Government of Canada bonds.

1. Introduction

A theory of the term structure of interest rates is intended to explain the relative pricing of default free bonds of different maturities. Complete theories of the term structure take as given the exogenous specifications of the economy: tastes, endowments, productive opportunities, and beliefs about possible future states of the world; then the prices of default free bonds of different maturities are derived from these exogenous specifications.[1] However, most extant theories of the term structure are partial equilibrium in nature and take as given beliefs about future realizations of the spot rate of interest, which are combined with simple assumptions about tastes to derive yields to maturity on discount bonds of different maturities.

The theory of the term structure has been cast traditionally in terms of the relationship between the forward rates which are inherent in the term structure and the corresponding expected future spot rates of interest. Thus the typical version of the pure expectations hypothesis asserts that forward rates are equal to expected future spot rates.[2] In contrast to the pure expectations hypothesis stands the liquidity premium hypothesis which

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[1] For example, Stiglitz (1970), Rubinstein (1976), and Roll (1970).

[2] It is now realized that this assumption is incompatible with universal risk neutrality, the assumption on which this version of the pure expectations hypothesis is usually based. See Merton (1973), Brennan and Schwartz (1977), Cox, Ingersoll and Ross (1977).

asserts that forward rates always exceed the corresponding expected future spot rates by a liquidity premium, which is required to compensate investors for the greater capital risk inherent in longer-term bonds. The market segmentation hypothesis can be regarded as a modification of the liquidity premium hypothesis to allow for positive or negative liquidity premia on longer-term bonds: this hypothesis recognizes that long-term bonds are not necessarily more risky than short-term bonds for investors who have long-term horizons, so that the prices of bonds of different maturities are determined by the preferences of investors with different horizons, with the result that forward rates may bear no systematic relationship to expected future spot rates. A major limitation of both liquidity premium and market segmentation hypotheses is their lack of specificity: since the relationship of liquidity premium to maturity is not specified, there are as many undetermined parameters in the model as there are bond maturities considered.

More recently it has been recognized that, if assumptions are made about the stochastic evolution of the instantaneous rate of interest in a continuous time model, much richer theories of bond pricing can be derived, which constrain the relationship between the risk premia on bonds of different maturities. Thus Merton (1973), Brennan and Schwartz (1977), and Vasicek (1976) have all assumed that the instantaneous spot rate of interest follows a Gauss–Wiener process. Then the arbitrage arguments, which are familiar from the option pricing literature, may be adduced to show that the prices of riskless bonds of all maturities must obey the same partial differential equation which contains only a single utility-dependent function. Since the whole term structure may be derived by solution of this partial differential equation, it follows that the liquidity premia for all maturities must depend upon this single function.

A significant deficiency of this arbitrage model of the term structure is the unrealism of the assumption about the stochastic process for the interest rate. It is assumed that since the instantaneous interest rate follows a Markov process, all that is known about future interest rates is impounded in the current instantaneous interest rate, so that the value of a default free bond of any maturity may be written as a function of this instantaneous interest rate and time. This implies that, apart from deterministic shifts over time in tastes, the whole term structure of interest rates may be inferred from the current instantaneous interest rate. This is clearly at odds with reality.

In this paper we take a step towards a more realistic approach to the relative pricing of bonds of different maturities by allowing changes in the instantaneous interest rate to depend not only on its current value but also on the long-term rate of interest, so that the long-term rate and the instantaneous rate follow a joint Gauss–Markov process. This expansion of the state space from one rate of interest to two is intended to reflect the assumption, which is the basis of both the pure expectations hypothesis and the liquidity premium hypothesis, that the current long-term rate of interest contains information about future values of the spot rate of interest. It should be clear that the model developed here, viewed simply as a model of the term structure, is less ambitious than the single state variable models referred to above: where they derive the long-term rate of interest, we take it as exogenous and attempt to explain only the intermediate portion of the yield curve in terms of its extremities. On the other hand, we avoid the objectionable implication of the above models that the long rate is a deterministic function of the current instantaneous interest rate. It is anticipated that the major contribution of the model developed here will be for the pricing of interest dependent contingent claims which contain an option element, such as savings bonds, retractable bonds and callable bonds. Then, just as the original Black–Scholes (1973) model determines the price of a call option in terms of the price of the underlying stock, without considering how the price of the underlying stock itself is determined, this model will permit the pricing of interest dependent claims in terms of the two exogenously given interest rates. However, before advancing to the more ambitious task of pricing bonds with an option element, it is useful to evaluate the ability of the model to price straight bonds of different maturities and this is the major objective of this paper: a subsidiary task is the estimation of the utility dependent function in the partial differential equation.

In two contemporaneous papers Richard (1976), and Cox, Ingersoll and Ross (1977) have also developed models of the term structure which incorporate two state variables. While our model takes these as the instantaneous rate and the long-term rate, their models take the state variables as the instantaneous real rate of interest and the rate of inflation, changes in which are assumed to be independent: from these state variables they are able to derive the long-term rate of interest. The advantage of their models then lies in the endogeneity of the long-term rate of interest, but this is obtained at the cost of introducing two utility dependent functions into the partial differential equation for bond prices, which considerably complicates the problems of empirical estimation. Our model avoids the need for one of the utility dependent functions by taking as the second state variable the long-term rate of interest which is inversely proportional to an asset price, the price of the consol bond: the risk associated with this state variable may then be hedged away. Both Richard and Cox, Ingersoll and Ross avoid the estimation problems posed by the two utility dependent functions in the partial differential equation by making explicit assumptions about the tastes of the representative investor: Richard considers both linear and logarithmic utility functions while Cox, Ingersoll and Ross consider only the logarithmic case. We assume that the utility dependent functions are constants and estimate their values from the data at hand.

In the following section the partial differential equation which must be satisfied by the value of any default free discount bond is derived. In section 3 the parameters of the assumed stochastic process for interest rates are estimated using data on Canadian interest rates. Section 4 reports the results of using the model to price a sample of Government of Canada bonds.

2. The pricing equation for discount bonds

Letting r denote the instantaneous rate of interest and l the long-term rate of interest which is taken as the yield on a consol bond which pays coupons continuously, it is assumed that r and l follow a joint stochastic process of the general type,

$$\begin{aligned} dr &= \beta_1(r,l,t)dt + \eta_1(r,l,t)dz_1, \\ dl &= \beta_2(r,l,t)dt + \eta_2(r,l,t)dz_2, \end{aligned} \tag{1}$$

where t denotes calendar time and dz_1 and dz_2 are Wiener processes with $E[dz_1]=E[dz_2]=0$, $dz_1^2=dz_2^2=dt$, $dz_1dz_2=\rho dt$. $\beta_1(.)$ and $\beta_2(.)$ are the expected instantaneous rates of change in the instantaneous and long-term rates of interest respectively, while $\eta_1^2(.)$ and $\eta_2^2(.)$ are the instantaneous variance rates of the changes in the two interest rates. ρ is the instantaneous correlation between the unanticipated changes in the two interest rates. Equation system (1) describes a situation in which changes in the instantaneous and long-term rates of interest are partially interdependent: both the expected change and the variance of the change in each interest rate may depend on the value of the other interest rate as well as on its own value. It is reasonable to suppose that the expected change in the instantaneous rate of interest will depend on the long-term rate of interest insofar as the long-term rate carries information about future values of the instantaneous rate; further, the expected change in the long rate must also depend on the current instantaneous rate if the expected rate of return on consol bonds is to be related to the rate of return on instantaneously riskless securities. In addition, (1) allows the unanticipated changes in the two interest rates to be correlated. While the degree of correlation is an empirical matter which will be addressed below, one may envisage the instantaneous rate changing as expectations of the instantaneous rate of inflation change, while the long rate responds to changing expectations about the long-run rate of inflation: it seems reasonable to suppose that changes in these expectations will be correlated but not perfectly so.

The price of a default free discount bond promising $1 at maturity is assumed to be a function of the current values of the interest rates, r and l, and time to maturity, τ, which we write as $B(r,l,\tau)$. Applying Itô's Lemma, the stochastic process for the price of a discount bond is

$$\mathrm{d}B/B=\mu(r,l,\tau)\mathrm{d}t+s_1(r,l,\tau)\mathrm{d}z_1+s_2(r,l,\tau)\mathrm{d}z_2, \tag{2}$$

where

$$\mu(r,l,\tau)=(B_1\beta_1+B_2\beta_2+\tfrac{1}{2}B_{11}\eta_1^2+\tfrac{1}{2}B_{22}\eta_2^2+B_{12}\rho\eta_1\eta_2-B_3)/B,$$

$$s_1(r,l,\tau)=B_1\,\eta_1/B,$$

$$s_2(r,l,\tau)=B_2\,\eta_2/B,$$

and

$$B_1=\partial B/\partial r,\quad B_2=\partial B/\partial l,\quad B_3=\partial B/\partial \tau \quad \text{etc.}$$

To derive the equilibrium relationship between expected returns on bonds of different maturities, consider forming a portfolio, P, by investing amounts x_1, x_2, x_3 in bonds of maturity τ_1, τ_2, τ_3 respectively. The rate of return on this portfolio is[3]

$$\begin{aligned}\mathrm{d}P/P=&[x_1\mu(\tau_1)+x_2\mu(\tau_2)+x_3\mu(\tau_3)]\mathrm{d}t\\&+[x_1s_1(\tau_1)+x_2s_1(\tau_2)+x_3s_1(\tau_3)]\mathrm{d}z_1\\&+[x_1s_2(\tau_1)+x_2s_2(\tau_2)+x_3s_2(\tau_3)]\mathrm{d}z_2.\end{aligned} \tag{3}$$

The rate of return on the portfolio will be non-stochastic if the portfolio proportions are chosen so that the coefficients of $\mathrm{d}z_1$ and $\mathrm{d}z_2$ in (3) are zero. That is, so that

$$\begin{aligned}&x_1s_1(\tau_1)+x_2s_1(\tau_2)+x_3s_1(\tau_3)=0,\\&x_1s_2(\tau_1)+x_2s_2(\tau_2)+x_3s_2(\tau_3)=0.\end{aligned} \tag{4}$$

Then, to avoid the possibility of arbitrage profits, it is necessary that the rate of return on this portfolio be equal to the instantaneous riskless rate of interest, r, so that

$$x_1(\mu(\tau_1)-r)+x_2(\mu(\tau_2)-r)+x_3(\mu(\tau_3)-r)=0. \tag{5}$$

The zero risk conditions (4) and the no arbitrage condition (5) constitute a set of three linear homogeneous equations in the three portfolio proportions. They will possess a solution if and only if

$$\mu(\tau)-r=\lambda_1(r,l,t)s_1(\tau)+\lambda_2(r,l,t)s_2(\tau), \tag{6}$$

[3]The arguments, r and l, are omitted from the functions $\mu(\cdot)$, $s_1(\cdot)$ and $s_2(\cdot)$ for the sake of brevity; they are to be understood.

where the functions $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ are independent of maturity, τ. Eq. (6) is an equilibrium relationship which constrains the relative risk premia on bonds of different maturities. It expresses the instantaneous risk premium on a discount bond of any maturity as the sum of two elements: these are proportional to the partial covariances of the bond's rate of return with the unanticipated changes in the instantaneous and long term rates of interest, $s_1(\cdot)$ and $s_2(\cdot)$ respectively. $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$ may then be regarded as the market prices of instantaneous and long term interest rate risk and will depend upon the utility functions of market participants. If the expressions for $\mu(\cdot)$, $s_1(\cdot)$ and $s_2(\cdot)$ are substituted in (6) the result will be a partial differential equation for the price of a discount bond, $B(r,l,\tau)$, which will contain the two utility dependent functions $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$.[4] However, by making use of the fact that l is a function of the price of an asset which we assume to be traded, a consol bond, it can be shown[5] that $\lambda_2(\cdot)$ is given by

$$\lambda_2(r,l,t) = -\eta_2/l + (\beta_2 - l^2 + rl)/\eta_2. \tag{7}$$

Eq. (7) expresses $\lambda_2(\cdot)$ in terms of the two rates of interest and the parameters of the stochastic process for the long-term rate of interest. It therefore enables us to eliminate this utility dependent function from the partial differential equation for the price of a discount bond, so that substitution in the equilibrium relationship (6) of the expressions for $\mu(\tau)$, $s_1(\tau)$ and $s_2(\tau)$, and use of eq. (7) to eliminate $\lambda_2(\cdot)$, permits us to re-write the equilibrium relationship (6) as the partial differential equation

$$\begin{aligned} &\tfrac{1}{2}B_{11}\eta_1^2 + B_{12}\rho\eta_1\eta_2 + \tfrac{1}{2}B_{22}\eta_2^2 \\ &+ B_1(\beta_1 - \lambda_1\eta_1) + B_2(\eta_2^2/l^2 + l^2 - rl) - B_3 - Br = 0. \end{aligned} \tag{8}$$

Given the stochastic process (1) for the two interest rates r and l, (8) is the basic partial differential equation for the pricing of default free discount bonds. This equation, together with the boundary condition specifying the payment to be received at maturity, say $B(r,l,0)=1$, may be solved to yield the prices of discount bonds of all maturities from which the whole term structure of interest rates may be inferred. The term structure at any point in time will depend upon the current values of the state variables r and l, as well as upon the unknown function $\lambda_1(\cdot)$. The prices of regular coupon bonds may be obtained by treating them as portfolios of discount bonds; alternatively, if coupons are paid continuously at the rate c, then c should be added to the left-hand side of the partial differential eq. (8). In addition, this

[4] This would be identical to the partial differential equation obtained by Richard (1976) if the variable l is interpreted as the rate of inflation rather than as the long term rate of interest.

[5] See appendix.

equation is valid for all types of default free interest dependent claims, so that it may be applied for example to the pricing of saving bonds or callable bonds by the introduction of the appropriate boundary conditions defining the payoffs on the claims.

It is interesting to note that the partial differential equation is not only independent of $\lambda_2(\cdot)$, the market price of long-term interest rate risk, it is also independent of $\beta_2(\cdot)$, the drift parameter for the long term interest rate, so that the solution is independent of the expected rate of return on the consol bond. This result is analogous to the finding within the simple Black–Scholes (1973) model for the pricing of stock options that the function expressing the equilibrium price of the option in terms of the price of the underlying stock is independent of the expected rate of return on the underlying stock. The reason for the two results is the same: there exists an asset for which the partial derivatives of its value with respect to all of the state variables is known: in this case the consol bond, and in the Black–Scholes case the stock. It can be shown that in general the number of unknown utility dependent parameters left in the partial differential equation will be equal to the number of state variables, excluding time, less the number of assets for which the partial derivatives of the value function are known: in the Black–Scholes case this is zero and in the present case it is one. The time variable is excluded since the pure reward for the passage of time is equal to the interest rate. This proposition is illustrated more formally in the appendix.

The coefficients of the partial differential eq. (8) are the utility dependent function, $\lambda_1(\cdot)$, and the parameters of the underlying stochastic process for the two interest rates, (1). Empirical application of the model requires that the parameters of this stochastic process be estimated and this is taken up in the next section.

3. Estimation of the stochastic process

3.1. *The form of the stochastic process*

Estimation of the stochastic process for interest rates (1) presupposes some stronger assumptions about the form of the process than we have made hitherto. The first restriction comes from the requirement that the excess of the expected rate of return on the consol bond over the instantaneous rate of interest be commensurate with the degree of long-term interest rate risk of the consol. This requirement is expressed in eq. (7): solving this equation for $\beta_2(\cdot)$, we find

$$\beta_2(r,l,t)=l^2-rl+\eta_2^2/l+\lambda_2\eta_2. \tag{9}$$

For empirical tractability it is assumed that $\lambda_2(\cdot)$, the market price of long-term interest rate risk, is constant.

The only other a priori restrictions which can be imposed on the stochastic process derive from the requirement that dominance by money be avoided, so that neither of the interest rates can be allowed to become negative. This possibility is avoided by assuming that

$$\eta_1(r,l,t)=r\sigma_1, \qquad \eta_2(r,l,t)=l\sigma_2, \tag{10}$$

and requiring that

$$\beta_1(r,l,t)\geqq 0. \tag{11}$$

Eqs. (10) and (11) jointly imply that $\beta_2(r,l,t)\geqq 0$. Eq. (10) specifies that the standard deviation of the instantaneous change in each interest rate is proportional to its current level.

To reflect the premise that the long-term rate contains information about future values of the instantaneous rate, it is assumed that the instantaneous rate stochastically regresses towards a function of the current long-term rate. This assumption and conditions (10) and (11) are satisfied by taking as the stochastic process for the logarithm of the instantaneous rate

$$\mathrm{d}\ln r=\alpha[\ln l-\ln p-\ln r]\,\mathrm{d}t+\sigma_1\mathrm{d}z_1, \tag{12}$$

which is equivalent to the assumptions that

$$\beta_1(r,l,t)=r[\alpha\ln(l/pr)+\tfrac{1}{2}\sigma_1^2], \tag{13}$$

$$\eta_1(r,l,t)=r\sigma_1. \tag{14}$$

The coefficient α represents the speed of adjustment of the logarithm of the instantaneous rate towards its current target value, $\ln(l/p)$, and p is a parameter relating the target value of $\ln r$ to the current value of $\ln l$.

Finally substituting for $\beta_2(\cdot)$ and $\eta_2(\cdot)$ from (9) and (10) in eq. (1), the stochastic process for the long-term rate of interest is

$$\mathrm{d}l=l[l-r+\sigma_2^2+\lambda_2\sigma_2]\,\mathrm{d}t+l\sigma_2\mathrm{d}z_2. \tag{15}$$

3.2. *The linearized form of the stochastic process*

Eqs. (12) and (15) constitute a non-linear system of stochastic differential equations governing the behaviour of the two interest rates. In order to estimate the system it is necessary first to linearize it, and to this end we approximate l and r by linear functions of $\ln l$ and $\ln r$. Thus, writing l and r

as functions of $\ln l$ and $\ln r$, and expanding in Taylor series about the mean sample values, $e^{\overline{\ln l}}$ and $e^{\overline{\ln r}}$,

$$\begin{aligned} l-r &= e^{\overline{\ln l}} - e^{\overline{\ln r}} \\ &\approx e^{\overline{\ln l}}(1-\overline{\ln l}) - e^{\overline{\ln r}}(1-\overline{\ln r}) + e^{\overline{\ln l}}\ln l - e^{\overline{\ln r}}\ln r. \end{aligned} \tag{16}$$

Then using Itô's Lemma to obtain the stochastic process for $\ln l$ from (15), and substituting for $(l-r)$ from (16), the linearized stochastic differential equation for the logarithm of the long-term rate may be written as

$$\mathrm{d}\ln l = [q - k_1 \ln r + k_2 \ln l]\mathrm{d}t + \sigma_2 \mathrm{d}z_2, \tag{17}$$

where

$$q = e^{\overline{\ln l}}(1-\overline{\ln l}) - e^{\overline{\ln r}}(1-\overline{\ln r}) + \tfrac{1}{2}\sigma_2^2 + \lambda_2\sigma_2,$$

while we may write the stochastic differential equation for the logarithm of the instantaneous rate as

$$\mathrm{d}\ln r = \alpha[\ln l - \ln r - \ln p]\mathrm{d}t + \sigma_1 \mathrm{d}z_1. \tag{18}$$

This linearized system of stochastic differential equations for the logarithms of the two interest rates is written in matrix notation as

$$\mathrm{d}y(t) = Ay(t)\mathrm{d}t + b\mathrm{d}t + \mathrm{d}\xi(t), \tag{19}$$

where

$$y(t) = \begin{pmatrix} \ln r(t) \\ \ln l(t) \end{pmatrix}, \qquad d\xi(t) = \begin{pmatrix} \sigma_1 \ \mathrm{d}z_1(t) \\ \sigma_2 \ \mathrm{d}z_2(t) \end{pmatrix},$$

$$A = \begin{pmatrix} -\alpha & \alpha \\ -k_1 & k_2 \end{pmatrix}, \qquad b = \begin{pmatrix} -\alpha \ln p \\ q \end{pmatrix}.$$

3.3. The exact discrete model

While (19) is a system of linear stochastic differential equations, the data on interest rates which are required to estimate it are available only at discrete intervals. One approach to estimation when there are prior restrictions on the parameters[6] has been proposed by Bergstrom (1966). This involves first substituting finite differences for differentials and averages of beginning and end of period values for the time dated vector $y(t)$, and then

[6] k_1 and k_2 are known and the coefficients of the two variables in the first equation are known to be equal in magnitude but opposite signs.

estimating the resulting linear equations by standard simultaneous equations methods. Unfortunately, as Phillips (1972) points out, the undesirable feature of this approach is the specification error which causes the resulting parameter estimates to be asymptotically biased. A more efficient and elegant procedure is to obtain the exact discrete model corresponding to (19) and to estimate the parameters from this model.

The exact discrete model corresponding to (19) is[7]

$$y(t) = e^{A} y(t-1) + A^{-1}[e^{A} - I]b + \zeta(t), \tag{20}$$

where

$$\zeta(t) = \int_{t-1}^{t} e^{(t-s)A} \, \mathrm{d}\xi(s),$$

and the variance covariance matrix of errors is

$$E[\zeta(t)\zeta'(t)] = \int_{0}^{1} e^{sA} \Sigma e^{sA'} \, \mathrm{d}s, \tag{21}$$

where Σ is the instantaneous variance covariance matrix with elements σ_1^2, σ_2^2, $\rho\sigma_1\sigma_2$.[8]

The matrix e^{A} is defined by

$$e^{A} \equiv T e^{\Lambda} T^{-1}, \tag{22}$$

where

$$e^{\Lambda} = \begin{pmatrix} e^{v_1} & 0 \\ 0 & e^{v_2} \end{pmatrix},$$

and v_1 and v_2 are the characteristic roots of the matrix A, while T is the matrix of characteristic vectors. In this case the characteristic roots are

$$\begin{aligned} v_1 &= (k_2 - \alpha + \sqrt{(k_2-\alpha)^2 - 4(k_1 - k_2)})/2, \\ v_2 &= (k_2 - \alpha - \sqrt{(k_2-\alpha)^2 - 4(k_1 - k_2)})/2, \end{aligned} \tag{23}$$

and the matrix of characteristic vectors is

$$T \equiv \begin{pmatrix} 1 & 1 \\ \dfrac{k_1}{k_2 - v_1} & \dfrac{k_1}{k_2 - v_2} \end{pmatrix}. \tag{24}$$

[7] See Bergstrom (1966), Phillips (1972), Wymer (1972).

[8] It can be shown that $\int_0^1 e^{sA} \Sigma e^{sA'} \, \mathrm{d}s \approx \Sigma$, so that the variance covariance matrix of errors from (20) provides good estimates of σ_1, σ_2 and $\sigma_{12} = \rho\sigma_1\sigma_2$.

Inverting A and carrying out the appropriate matrix multiplications in (20) the exact discrete model to be estimated is

$$\begin{aligned} y_1(t) = {} & \frac{1}{v_1 - v_2}[e^{v_2}(k_2 - v_2) - e^{v_1}(k_2 - v_1)]\, y_1(t-1) \\ & + \frac{1}{k_1(v_1 - v_2)}[(k_2 - v_1)(k_2 - v_2)(e^{v_1} - e^{v_2})]\, y_2(t-1) \\ & + \frac{1}{(k_1 - k_2)(v_1 - v_2)}\{\ln p[k_2[e^{v_1}(k_2 - v_1) - e^{v_2}(k_2 - v_2) + v_1 - v_2] \\ & - \alpha k_1(e^{v_1} - e^{v_2})] + q[k_2(k_2 - v_1)(k_2 - v_2)(e^{v_1} - e^{v_2})/\alpha k_1 \\ & - e^{v_1}(k_2 - v_2) + e^{v_2}(k_2 - v_1) + v_1 - v_2]\} + \zeta_1(t). \end{aligned} \tag{25}$$

$$\begin{aligned} y_2(t) = {} & \frac{k_1}{v_1 - v_2}(e^{v_2} - e^{v_1})\, y_1(t-1) + \frac{1}{v_1 - v_2}[e^{v_1}(k_2 - v_2) - e^{v_2}(k_2 - v_1)]\, y_2(t-1) \\ & + \frac{1}{(k_1 - k_2)(v_1 - v_2)}\{\ln p[k_1[e^{v_1}(k_2 - v_1) - e^{v_2}(k_2 - v_2) + v_1 - v_2] \\ & - \alpha k_1(e^{v_1} - e^{v_2})] + q[(k_2 - v_1)(k_2 - v_2)(e^{v_1} - e^{v_2})/\alpha \\ & - e^{v_1}(k_2 - v_2) + e^{v_2}(k_2 - v_1) + v_1 - v_2]\} + \zeta_2(t). \end{aligned} \tag{26}$$

Summarizing the analysis to this point, the system of stochastic differential equations (1) was first specialized by assuming that the standard deviation of the unanticipated instantaneous changes in each interest rate is proportional to the current level of that rate (10); by requiring that the instantaneous expected rate of return on a consol bond be commensurate with its degree of long-term interest rate risk (9), where $\lambda_2(\cdot)$, the market price of long-term interest rate risk, is taken as constant; and by requiring that the logarithm of the instantaneous interest rate stochastically regress towards a target value which depends on the current value of the long-term rate (12). The resulting system of stochastic differential equations, (12) and (15), was then linearized to yield the system (19), where $y_1(t)$ and $y_2(t)$ are the logarithms of the instantaneous rate and the long-term rate respectively. Finally, since the equation system is to be estimated using data on r and l at discrete time intervals, the exact discrete model, (25) and (26), corresponding to the linearized form (19) was found.

3.4. Empirical results

The three coefficients of the equation system (25), (26) to be estimated are α, $\ln p$ and q. In addition we require an estimate of the variance–covariance matrix Σ, since the elements of this matrix appear as coefficients in the partial differential eq. (8) for the value of a bond. The estimation was carried out using a non-linear procedure described by Malinvaud (1966) and employed by Phillips (1972) in a similar context. The data for the instantaneous rate of interest were the yields on 30-day Canadian Bankers' Acceptances converted to an equivalent continuously compounded annual rate of interest, while the long-term rate of interest was the continuously compounded equivalent of the average yields to maturity on Government of Canada bonds with maturities in excess of 10 years. Both interest rates series are mid-market closing rates on the last Wednesday of each month from January 1964 to December 1976.[9]

The estimated equation system is

$$\begin{aligned} \mathrm{d}\ln r &= \underset{(0.0050)}{0.0701}[\ln l/r - \underset{(0.0050)}{0.0599}]dt + 0.0736\mathrm{d}z_1, \\ \mathrm{d}\ln l &= [\underset{(0.0020)}{0.0060} - 0.0051\ln r + 0.0058\ln l]\mathrm{d}t + 0.0250\mathrm{d}z_2, \end{aligned}$$

where the standard errors of the estimated coefficient are in parentheses and the coefficients of $-\ln r$ and $\ln l$ are the computed values of k_1 and k_2. The estimated correlation between the errors in the two equations, ρ, is 0.3747, and the adjustment coefficient of 0.0701 in the first equation implies that half of the adjustment in the instantaneous rate occurs within 10 months.

In terms of the coefficients of the basic partial differential eq. (8) for the pricing of discount bonds, the parameter estimates imply

$$\begin{aligned} &\eta_1 \equiv r\sigma_1 = 0.0736\,r,\quad \eta_2 \equiv l\sigma_2 = 0.0250\,l,\quad \rho = 0.3747, \\ &\beta_1(r,l,t) = r[\alpha\ln(l/pr) + \tfrac{1}{2}\sigma_1^2] \\ &\qquad = r[0.0701(\ln l/r - 0.0599) + \tfrac{1}{2}(0.0736^2)]. \end{aligned}$$

4. Bond pricing and the term structure of interest rates

Re-writing the partial differential eq. (8) to take account of the specific stochastic process for r and l assumed in the previous section, we have, substituting for $\beta_1(\cdot)$, $\eta_1(\cdot)$ and $\eta_2(\cdot)$,

$$\begin{aligned} &\tfrac{1}{2}B_{11}r^2\sigma_1^2 + B_{12}rl\rho\sigma_1\sigma_2 + \tfrac{1}{2}B_{22}l^2\sigma_2^2 \\ &+ B_1 r[\alpha\ln(l/pr) + \tfrac{1}{2}\sigma_1^2 - \lambda_1\sigma_1] + B_2 l[\sigma_2^2 + l - r] - B_3 - Br = 0. \end{aligned} \tag{27}$$

[9] Taken from the Bank of Canada Review, Cansim Series 2560.33 and 2560.13.

Then the value of a discount bond promising \$1 at maturity, $\tau=0$, is given by the solution to eq. (27) subject to the boundary condition

$$B(r,l,0)=1. \tag{28}$$

Using the values of α, $\ln p$, ρ, σ_1 and σ_2 estimated in section 3, eq. (27) with boundary condition (28) was solved[10] for values of λ_1, the market price of instantaneous interest rate risk, of -0.04, 0.0, 0.09. The resulting values of $B(r,l,\tau)$ are present value factors: for a given value of $\lambda_1, B(r,l,\tau)$ is the present value of \$1 payable with certainty in τ periods when the instantaneous and long-term rates of interest are r and l respectively.

A sample of 101 Government of Canada bonds was priced using the present value factors computed for each of the three values of λ_1. The bonds were priced on the last Wednesday of each quarter from January 1964 to January 1977 by applying the present value factors appropriate to the prevailing instantaneous and long-term rates of interest to the promised coupon and principal payments for each bond. The sample includes all Government of Canada bonds with maturities less than 10 years for which prices were available in the Bank of Canada Quarterly Review and which were neither callable nor exchangeable. The root mean square price prediction error was calculated for each of the three values of λ_1, and quadratic interpolation was used to estimate the value of λ_1 which minimizes the root mean square prediction error.[11] This estimated value of λ_1 was 0.0355 and the bonds were then priced for this value of λ_1.

In addition, for each of the four values of λ_1, yields to maturity were calculated based on the predicted bond values each quarter and these predicted yields to maturity were compared with the actual yields to maturity. The comparison of actual and predicted bond values and yields to maturity is reported, for each value of λ_1, in table 1: in this table all bonds are treated as having a par value of 100. Thus for the estimated value of $\lambda_1 = 0.0355$, the root mean square prediction error for bond prices is 1.56 and the mean error is -0.17. For the same value of λ_1, the root mean square prediction error for yields to maturity is 0.67% and the mean error is 0.24%. It is to be anticipated that the model will be less successful in predicting yields to maturity than in predicting bond prices, since a small error in the predicted bond price will cause a very large error in the predicted yield to maturity for short dated bonds.

[10]The solution procedure is described in the appendix.

[11]That is, a quadratic curve was fitted to the three pairs of RMSE and λ_1, and the RMSE minimizing value of λ_1 was computed. When the bonds were priced using this value of λ_1 the RMSE agreed with the interpolated value. This non-linear estimation procedure leads to a maximum likelihood estimator under the usual assumption of normal, independent, homoscedastic errors. A more efficient estimator which would allow for a generalized error structure was contemplated but ruled out on the basis of computational cost.

For both bond prices and yields to maturity, the actual values were regressed on the predicted values and the resulting regression statistics are reported in table 1 also. For unbiased predictions the intercept term (α) should be zero, and the slope coefficient (β) should be equal to unity. The actual slope coefficients for $\lambda_1 = 0.0355$ are 0.93 for bond prices and 0.79 for yields to maturity. While these regression results should be treated with caution since there is no assurance that the errors are either independent or normally distributed, it is encouraging to observe that there is a strong, though certainly not perfect, correspondence between the actual and predicted values.

Tables 2 and 3 report the results of predicting bond values and yields to maturity for the last Wednesday of each January from 1964 to 1977. These results are representative of those obtained for the other quarters for which predictions were made. While there is reasonable stability in the relationship between actual and predicted bond values, the relationship between actual and predicted yields to maturity is much more erratic. This reflects the greater difficulty in predicting this variable, referred to above, and also suggests that there are factors which are not encompassed in our model which determine the shape of the term structure.

Table 1

Predicted and actual bond prices and yields to maturity for alternative values of λ_1 (t-ratios in parentheses).

	Values of λ_1			
	-0.04	0.0	0.0355	0.09
Bond prices				
RMSE	1.95	1.65	1.56	1.74
Mean error	-1.05	-0.59	-0.17	0.41
α	13.44 (21.44)	10.01 (15.40)	7.28 (10.46)	4.12 (5.04)
β	0.87 (134.04)	0.90 (134.44)	0.93 (129.57)	0.95 (114.23)
R^2	0.93	0.93	0.93	0.91
Yields to maturity				
RMSE (%)	0.81	0.72	0.67	0.64
Mean error (%)	0.52	0.37	0.24	0.06
α (%)	1.12 (18.40)	1.15 (18.24)	1.18 (18.23)	1.25 (18.40)
β	0.77 (90.63)	0.78 (87.61)	0.79 (84.44)	0.80 (79.56)
R^2	0.86	0.86	0.85	0.83

Table 2

Predicted and actual bond prices by period for $\lambda_1 = 0.0355$ (α, β are the coefficients from the regression of actual values on predicted values).

Year (last Wednesday of January)	No. of observations	RMSE	Mean error (pred.-actual)	α (*T-stat.*)	β (*T-stat*)	R^2	Instantaneous interest rate	Long-term interest rate
1964	17	1.09	0.61	4.62 (0.43)	0.95 (8.68)	0.82	3.69%	5.17%
1965	16	1.04	0.39	23.29 (2.91)	0.76 (9.59)	0.86	3.81	4.69
1966	20	2.15	1.63	37.75 (3.24)	0.61 (5.23)	0.59	4.00	5.41
1967	21	0.91	−0.58	9.49 (1.10)	0.91 (10.44)	0.84	5.85	5.60
1968	22	1.64	0.93	−37.57 (−4.60)	1.38 (16.38)	0.93	6.40	6.54
1969	28	0.97	0.46	−14.97 (−3.53)	1.15 (26.39)	0.96	6.60	7.16
1970	31	1.06	0.07	−10.98 (−5.52)	1.11 (53.39)	0.99	8.90	8.31
1971	30	2.01	−1.61	−0.45 (−0.10)	1.02 (22.66)	0.95	6.00	6.67
1972	32	1.36	0.43	−7.84 (−1.67)	1.07 (23.17)	0.95	3.95	6.73
1973	28	0.46	−0.06	−0.22 (−0.13)	1.00 (59.39)	0.99	4.75	7.16
1974	24	1.95	−1.84	2.42 (1.19)	0.99 (46.00)	0.99	8.75	7.75
1975	22	3.37	−2.84	17.04 (3.32)	0.85 (15.86)	0.92	7.00	8.30
1976	22	1.71	−1.44	5.32 (2.02)	0.96 (33.76)	0.98	9.00	9.29
1977	16	0.99	−0.94	3.70 (1.18)	0.97 (30.88)	0.98	8.33	9.09

Table 3

Predicted and actual yields to maturity by period for $\lambda_1 = 0.0355$ (α, β are the coefficients from regression of actual values on predicted values).

Year (last Wednesday of January)	No. of observations	RMSE	Mean error (pred. actual)	α (T-stat.)	β (T-stat.)	R^2	Instantaneous interest rate	Long-term interest rate
1964	17	0.21%	−0.16%	−1.56 (−2.86)	1.40 (11.05)	0.88	3.69%	5.17%
1965	16	0.25	0.01	−5.26 (−6.13)	2.22 (11.15)	0.89	3.81	4.69
1966	20	0.53	−0.50	−1.52 (−2.06)	1.44 (9.02)	0.81	4.00	5.41
1967	21	0.64	0.50	13.93 (10.38)	−1.57 (−6.58)	0.68	5.85	5.60
1968	22	0.39	−0.12	35.38 (5.09)	−4.54 (−4.16)	0.45	6.40	6.54
1969	28	0.24	−0.06	−27.45 (−8.36)	5.05 (10.44)	0.80	6.60	7.16
1970	31	0.79	0.42	16.66 (11.44)	−1.00 (−5.88)	0.54	8.90	8.31
1971	30	0.85	0.76	−18.70 (−4.40)	3.86 (5.70)	0.53	6.00	6.67
1972	32	0.35	−0.04	−0.66 (−0.95)	1.13 (8.69)	0.71	3.95	6.73
1973	28	0.20	0.05	−0.48 (−1.24)	1.07 (16.64)	0.91	4.75	7.16
1974	24	1.20	1.04	11.04 (14.58)	−0.51 (−5.37)	0.56	8.75	7.75
1975	22	1.11	1.05	2.52 (0.98)	0.53 (1.57)	0.10	7.00	8.30
1976	22	0.75	0.68	6.80 (0.88)	0.17 (0.198)	0.00	9.00	9.29
1977	16	0.52	0.49	7.55 (1.35)	0.04 (0.06)	0.00	8.33	9.09

One factor which has been neglected in the model developed in this paper is the role of income taxes and their differential impact on coupon income and capital gains. To test whether income taxes cause the coupon stream of a bond to be valued less highly than the principal repayment at maturity, the predicted value of the principal payment was subtracted from the actual bond price and the difference was regressed on the predicted value of the coupon stream. If income taxes are important in the pricing of bonds, the resulting regression coefficient should be less than unity, the difference between unity and the estimated regression coefficient measuring the effective tax rate on coupon income. The regression results are reported in table 4 for the different values of λ_1. The evidence presented in this table suggests that the effect of income taxes is slight: for the estimated value of $\lambda_1 = 0.0355$, the estimated tax rate is only 4%, and even for $\lambda_1 = 0.09$ the estimated tax rate is only 8%.

Table 4

The influence of taxes on bond prices: (bond price predicted value of principal) $= \alpha + \beta$ (predicted value of coupons).

	Values of λ_1			
	−0.04	0.0	0.0355	0.09
α	0.62 (8.03)	0.67 (9.11)	0.71 (10.04)	0.78 (11.28)
β	1.03 (249.33)	0.99 (254.09)	0.96 (256.38)	0.92 (256.24)
R^2	0.98	0.98	0.98	0.98

5. Conclusion

In this paper we have developed a theory of the term structure of interest rates based on the assumption that the value of all default free discount bonds may be written as a function of time and two interest rates, the instantaneous rate and the long-term rate, which follow a joint Markov process in continuous time. This assumption permitted us to derive in section 2 a partial differential equation which must be satisfied by the values of all default free discount bonds. The partial differential equation contains two utility dependent functions, $\lambda_1(\cdot)$ and $\lambda_2(\cdot)$, but $\lambda_2(\cdot)$ was eliminated by making use of the assumption that there exists a traded asset, a consol bond, which corresponds to one of the state variables, the long-term rate of interest.

In section 3 the stochastic process for the two interest rates was specialized and estimated using data on Canadian interest rates. The partial differential equation was then solved using the estimated parameters and selected values for the market price of instantaneous interest rate risk, λ_1, to find the value of λ_1 which minimized the price prediction errors for a sample of Canadian government bonds, and the predictive ability of the model was evaluated: the root mean square prediction error for bond prices was of the order of 1.5%.

It is anticipated that models of this type will have application in the management of bond portfolios and studies of the efficiency of bond markets. Perhaps the most interesting application is to the pricing of bonds which contain an option such as callable bonds and saving bonds. The latter are default free securities allowing the holder the right of redemption prior to maturity at a predetermined series of redemption prices. While instruments of this type are common in North America and several European countries, including France, Germany, Italy and the United Kingdom, they have received virtually no attention to date from financial economists. Work is currently in progress to apply the model developed in this paper to Canadian savings bonds.

This model should be seen as a first step in the application of a new approach to the term structure of interest rates and the pricing of default free securities. Further work is required on the specification and estimation of both the stochastic process for the interest rates and the market price of interest rate risk.

Appendix

A.1. The market price of long-term interest rate risk, $\lambda_2(r,l,t)$

It is shown here that if there exists a consol bond, the utility dependent market price of long-term interest rate risk may be expressed in terms of the two interest rates and the parameters of the stochastic process for the long-term rate of interest. Let $V(l)$ denote the price of a consol bond paying a continuous coupon at the rate of $1 per period. Then the long-term rate of interest is defined by

$$V(l)=l^{-1}, \tag{29}$$

so that, applying Itô's Lemma, the stochastic process for the price of a consol bond is

$$\mathrm{d}V/V=(\eta_2^2/l^2-\beta_2/l)\mathrm{d}t-(\eta_2/l)\mathrm{d}z_2. \tag{30}$$

Then, defining $s_1(\alpha)$ and $s_2(\alpha)$ as the partial covariances of the consol's bond's rate of return with the unanticipated changes in the two interest rates, it follows from eq. (30) that $s_1(\alpha)=0$, $s_2(\alpha)=-\eta_2/l$. Further, defining $\mu(\alpha)$

as the expected instantaneous rate of return on the consol bond including both the expected capital gain which is obtained from (30) and the rate of coupon payment per dollar of principal,

$$\mu(\alpha) = \eta_2^2/l^2 - \beta_2/l + l. \tag{31}$$

Now the expected rate of return on the consol bond must also satisfy the equilibrium risk premium equation, (6), so that substituting in this equation for $\mu(\alpha)$, $s_1(\alpha)$ and $s_2(\alpha)$ and solving for $\lambda_2(\cdot)$, we obtain

$$\lambda_2(r,l,t) = -\eta_2/l + (\beta_2 - l^2 + rl)/\eta_2, \tag{32}$$

which is eq. (7) of the text.

A.2. Asset prices and state variables

This section illustrates for eq. (6) that the number of utility dependent functions left in the partial differential equation is equal to the number of state variables, excluding time, less the number of assets for which the partial derivatives of the value functions are known. Substitute in the equilibrium condition (6) the expressions for $\mu(\cdot)$, $s_1(\cdot)$ and $s_2(\cdot)$ to obtain

$$\begin{aligned} &B_1\beta_1 + B_2\beta_2 + \tfrac{1}{2}B_{11}\eta_1^2 + \tfrac{1}{2}B_{22}\eta_2^2 + B_{12}\rho\eta_1\eta_2 - B_3 - rB \\ &= \lambda_1 B_1\eta_1 + \lambda_2 B_2\eta_2. \end{aligned} \tag{33}$$

Now suppose that there exists an asset with value G, all of whose partial derivatives with respect to the state variables are known. The value of the asset must also satisfy the same partial differential equation,

$$\begin{aligned} &G_1\beta_1 + G_2\beta_2 + \tfrac{1}{2}G_{11}\eta_1^2 G_{22}\eta_2^2 + G_{12}\rho\eta_1\eta_2 - G_3 - rG \\ &= \lambda_1 G_1\eta_1 + \lambda_2 G_2\eta_2. \end{aligned} \tag{34}$$

Then to eliminate λ_2 and β_2 eq. (34) is multiplied by B_2/G_2 and subtracted from (33) to yield

$$\begin{aligned} &(B_1 - B_2G_1/G_2)\beta_1 + \tfrac{1}{2}(B_{11} - B_2G_{11}/G_2)\eta_1^2 \\ &+ \tfrac{1}{2}(B_{22} - B_2G_{22}/G_2)\eta_2^2 + (B_{12} - B_2G_{12}/G_2)\rho\eta_1\eta_2 \\ &- (B_3 - B_2G_3/G_2) - r(B - B_2G/G_2) \\ &= \lambda_1(B_1 - B_2G_1/G_2)\eta_1. \end{aligned} \tag{35}$$

Since G and all of its partial derivatives are known functions, (35) contains only a single utility dependent function, $\lambda_1(\cdot)$, and the drift parameter for the corresponding state variable, β_1. If G is the consol bond, then substitution of the appropriate partial derivatives in (35) will yield our partial differential eq. (8). It should be clear that if G_1 were not zero, it would have been possible to eliminate λ_1 and β_1 instead of λ_2 and β_2, and that if a second distinct asset exists whose partial derivatives are known it will be possible to eliminate all four parameters.

A.3. Solution of the partial differential equation

Since there is no known analytic solution to the differential eq. (27) we apply a finite difference solution procedure. This requires that the equation be transformed to take advantage of the natural boundary conditions which occur as the interest rates approach zero and infinity.

To transform the equation, define the new state variables u_1 and u_2 where[12]

$$u_1 = 1/(1+nr), \quad u_2 = 1/(1+nl),$$

and let $B(r,l,\tau) \equiv b(u_1, u_2, \tau)$.

Writing the partial derivatives of $B(\cdot)$ in terms of those of $b(\cdot)$, we have

$$B_1 = -nu_1^2 b_1, \quad B_2 = -nu_2^2 b_2,$$
$$B_{11} = n^2 u_1^4 b_{11} + 2n^2 u_1^3 b_1, \quad B_{22} = n^2 u_2^4 b_{22} + 2n^2 u_2^3 b_2,$$
$$B_3 = b_3.$$

Substituting for r, l and the derivatives of $B(\cdot)$ in (27), we obtain the transformed equation

$$\begin{aligned} &\tfrac{1}{2} b_{11} u_1^2 (1-u_1^2)\sigma_1^2 + b_{12} u_1 u_2 (1-u_1)(1-u_2)\rho u_1 u_2 + \tfrac{1}{2} b_{22} u_2^2 (1-u_2^2)\sigma_2^2 \\ &+ b_1 u_1 (1-u_1)[\sigma_1^2(\tfrac{1}{2}-u_1) - \alpha(\ln u_1(1-u_2)/pu_2(1-u_1)) + \lambda_1\sigma_1] \\ &+ b_2 u_2 (1-u_2)[-\sigma_2^2 u_2 - (1-u_2)/nu_2 + (1-u_1)/nu_1] \\ &- b_3 - b(1-u_1)/nu_1 = 0. \end{aligned} \tag{36}$$

The solution to this differential equation must satisfy the maturity boundary condition which is defined by assuming that the bond pays \$1 at maturity:

[12] The parameter n was chosen so that approximately one half of the range of u_1 and u_2 (0, 1) relates to the relevant range of interest rates, 0–20%, in which solution accuracy is required, i.e. $n=40$.

$$b(u_1, u_2, 0) = 1. \tag{37}$$

In addition we have the following natural boundaries obtained by letting u_1, u_2 approach zero and one[13] in the differential eq. (36):

(i) For $r = \infty (u_1 = 0)$, $l = \infty (u_2 = 0)$.

Multiply (36) by nu_1 and let u_1 and u_2 approach zero to obtain

$$b(0, 0, \tau) = 0. \tag{38}$$

(ii) For $r = \infty (u_1 = 0)$, $l \neq \infty$.

Multiply (36) by nu_1 and let u_1 approach zero to obtain the ordinary differential equation

$$b_2(0, u_2, \tau) u_2 (1 - u_2) - b(0, u_2, \tau) = 0.$$

Solving this equation and imposing the requirement that $b(0, u_2, \tau) \leqq 1$, we have

$$b(0, u_2, \tau) = 0. \tag{39}$$

(iii) For $l = \infty (u_2 = 0)$, $r \neq \infty$.

Divide (36) by $\ln u_2$ and let u_2 approach zero to obtain

$$\alpha u_1 (1 - u_1) b_1 (u_1, 0, \tau) = 0,$$

and since $b(0, 0, \tau) = 0$ from (38), this implies that

$$b(u_1, 0, \tau) = 0. \tag{40}$$

The boundary conditions (38)–(40) state that if either interest rate is infinite, the value of the bond is zero.

(iv) For $r = 0$ $(u_1 = 1)$, $l = 0$ $(u_2 = 1)$.

Setting u_1 and u_2 equal to unity in (36),

$$b_3(1, 1, \tau) = 0.$$

Combining this with the maturity value boundary, (37), we obtain

$$b(1, 1, \tau) = 1. \tag{41}$$

[13]This corresponds to letting the interest rates r and l approach infinity and zero respectively.

(v) For $r=0$ $(u_1=1)$, $l\neq 0$.

Taking the limit in (36) as $u_1\to 1$,

$$\tfrac{1}{2}b_{22}u_2^2(1-u_2)^2\sigma_2^2+b_2u_2(1-u_2)[-\sigma_2^2u_2-(1-u_2)\ln u_2]-b_\tau=0. \tag{42}$$

$b(1,u_2,\tau)$ is obtained as the solution to (42) subject to the boundary conditions (37), (40) and (41).

(vi) For $l=0$ $(u_2=1)$, $r\neq 0$.

Divide (36) by $\ln(1-u_2)$ and let $u_2\to 1$ to obtain

$$\alpha u_1(1-u_1)b_1(u_1,1,\tau)=0 \tag{43}$$

The solution to (43) subject to the boundary (40) is

$$b(u_1,1,\tau)=1. \tag{44}$$

The finite difference approximation to (36) is obtained by defining $b(\cdot)$ at discrete intervals,

$$\begin{aligned} b(u_{1i},u_{2j},\tau_k) &\equiv b(hi,hj,gk) \\ &\equiv b_{i,j,k}, \qquad i,j=0,\ldots,m, \quad k=0,\ldots,K, \end{aligned} \tag{45}$$

where h and g are the step sizes for the interest rates and time to maturity respectively; since u_1 and u_2 are defined on the interval (0, 1), $hm=1$. Then writing finite differences in place of partial derivatives, (36) may be approximated by

$$\begin{aligned} &c_1^{i,j}b_{i-1,j-1,k}+c_2^{i,j}b_{i-1,j,k}+c_3^{i,j}b_{i-1,j+1,k} \\ &+c_4^{i,j}b_{i,j-1,k}+c_5^{i,j}b_{i,j,k}+c_6^{i,j}b_{i,j+1,k} \\ &+c_7^{i,j}b_{i+1,j-1,k}+c_8^{i,j}b_{i+1,j,k}+c_9^{i,j}b_{i+1,j+1,k} \\ &=b_{i,j,k+1}, \qquad i=1,\ldots,m-1, \quad j=1,\ldots,m-1, \end{aligned} \tag{46}$$

where $c_1^{i,j}$ etc. are coefficients derived from the parameters of the equation.

(46) is a system of $(m-1)^2$ equations in the $(m+1)^2$ unknowns $b_{i,j,k}$ $(i,j=0,1,\ldots,m)$; the remaining $4m$ equations are provided by the natural boundary conditions (i)–(vi) above.[14] The augmented system of equations may be solved recursively for the unknowns $b_{i,j,k}$ in terms of $b_{i,j,k+1}$, since the values

[14] Values of $b(1,u_2,\tau)$ are obtained by solving the finite difference approximation to (42).

$b_{i,j,0}$ are given by the maturity boundary condition (37). To take advantage of the structure of the coefficient matrix the equations were solved by the method of successive over-relaxation.[15]

[15]Westlake (1968).

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