by Michael J. Brennan and Eduardo S. Schwartz

Bond Pricing and Market Efficiency

The discount, or present value, function gives the present value of one dollar receivable in x periods as a function of the prevailing yield curve. Once the discount function is known, it can be used to value any default-free, straight bond with a predetermined coupon and maturity date.

If we assume that the yield curve is determined solely by its end-points (the short rate and the long, or consol, rate) and that the bond market is in equilibrium, so that there is no possibility of earning arbitrage profits, then the shape of the discount function will depend upon expectations about future values of the short and long interest rates and upon investor attitudes toward risk.

Expectations about interest rates were estimated using December 1958 to December 1979 interest rate and bond price data. Based on these data, the market price of risk was estimated by finding the value that yielded the best discount function—i.e., the one yielding bond prices closest to observed prices. Prices for the entire period were predicted on the basis of data estimated over the whole period; prices for the second half of the period were predicted on the basis of data estimated over the first half. The price prediction errors were then tested as predictors of subsequent bond returns. The results revealed a highly significant relation between the valuation error and the rate of return over the next (monthly or quarterly) time interval.

Apart from its ability to detect over and underpriced bonds, the model may be used in bond portfolio management to predict bond prices under alternative interest rate scenarios. It may also be applied to special features such as call provisions and to traded options on fixed-income securities.

THIS ARTICLE reports the results of applying an equilibrium model to the pricing of U.S. government bonds for the period 1958 to 1979, using data drawn from the CRSP Government Bond File. Comparing the model pricing errors with subsequent bond returns allows us to evaluate the ability of the model to detect underpriced and overpriced bonds. The data reveal a strong relation between price prediction errors and subsequent bond returns.

As well as extending our knowledge of the bond pricing process, the model permits us to adjust raw yields to maturity for coupon effects and to value bonds with call and retraction options, as well as traded options on bonds and Treasury bills. The model also provides the basis for assessing the risk of bond portfolios and for devising appropriate immunization strategies.[1]

1. Footnotes appear at end of article.

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Figure A The Yield Curve

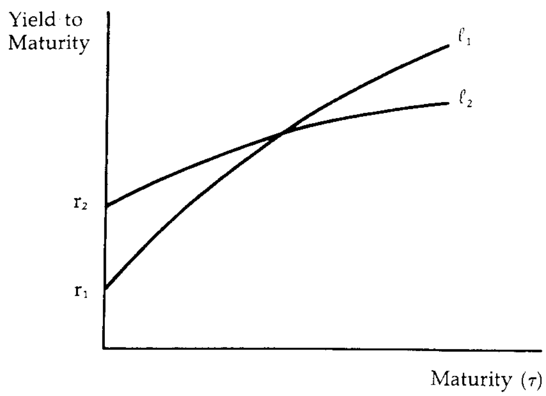
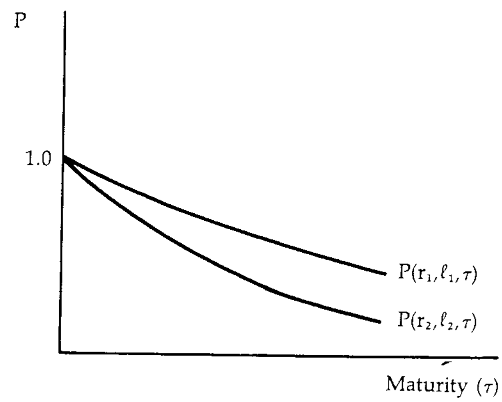


Figure B The Discount Function



The Bond Pricing Model[2]

The essential assumptions underlying the model are two—(1) that the yield curve is determined solely by its end-points and (2) that the market is in equilibrium, so that there is no possibility of earning arbitrage profits. In addition, the market is assumed to be perfect, so that taxes and transaction costs can be ignored and trading can take place continuously.

The Yield Curve Assumption: The interest rate on an immediately maturing riskless bond, r — the "short rate"—may be thought of as corresponding to a Treasury bill rate. The yield on a perpetuity or consol bond, ℓ —the "long rate" or "consol rate"—may in empirical application have to be approximated by the yield on a very long-term bond. The values of r and ℓ at any moment in time determine the two ends of the yield curve, as shown in Figure A. The yield curve assumption is that the whole yield curve is determined by the values of these two end-points, r and ℓ.

In fact, we shall find it convenient to deal, not with the yield curve directly, but with the *discount function*. The discount, or present value, function gives the present value of one dollar receivable in τ periods as a function of the prevailing yield curve as represented by the current values of r and ℓ. Thus $P(r,\ell,\tau)$ is the present value of one dollar receivable in τ periods, given that the short and long rates are r and ℓ, respectively.

Figure B presents the discount function for two different interest rate or yield curve scenarios. Whatever the yield curve, the value of the discount function is unity when τ equals zero: The present value of an immediate payment of one dollar is always one dollar. The value of the discount function decreases as the time to maturity of the payment increases, reflecting the time value of money.

The discount function is important because, once we know it, we can use it to value any default-free bond (provided that the bond is a straight bond with a predetermined coupon and maturity date). Table I illustrates this fact. Here we have assumed that r is eight per cent and ℓ is 10 per cent; under this interest rate scenario, the one and two-year discount factors, read off the discount function, are 0.917 and 0.819. The bond value is obtained by multiplying the promised payments by the appropriate discount factors and adding.

The Assumption of Equilibrium: We assume that the discount function is such that there is no possibility of forming a portfolio of bonds whose return is riskless unless its rate of return is equal to the current riskless interest rate, r.

Table I Using the Discount Function to Value a Coupon Bond
(*two-year bond*)

Year	1	2
	\$100	\$1,100

$r = 8\%$ $\ell = 10\%$
$P(8\%,10\%,1) = 0.917$
$P(8\%,10\%,2) = 0.819$

Year	Payment	×	Discount Factor	=	Discounted Value
1	\$ 100		0.917		\$ 91.70
2	\$1,100		0.819		\$900.70
			Bond Value	=	\$992.40

Given the yield curve assumption and the assumption of equilibrium, the shape of the discount function will depend upon expectations about future values of the interest rates r and ℓ and upon investor attitudes towards risk. The model is thus consistent with traditional liquidity premium theories of the term structure in which the yield curve depends upon expectations about future interest rates and upon a liquidity premium that reflects attitudes towards risk.

Changes in Interest Rates

Expectations about future values of r and ℓ are represented in this model by the parameters of the probabilistic or stochastic process, which is assumed to generate future values of r and ℓ, namely:

$$\tilde{\Delta}r = [a_1 + b_1(\ell - r)] + rS_r\tilde{\Delta}Z_r , \quad (1)$$

$$\tilde{\Delta}\ell = \ell(a_2 + b_2r + c_2\ell) + \ell S_\ell\tilde{\Delta}Z_\ell , \quad (2)$$

where $\tilde{\Delta}r$ and $\tilde{\Delta}\ell$ are the changes in the two interest rates. The unpredictable parts of the changes are represented by the terms involving $\tilde{\Delta}Z_r$ and $\tilde{\Delta}Z_\ell$, which are normally distributed random variables with mean zero and variance of one; the correlation between them is denoted by ρ Since these terms are multiplied by r and ℓ, this formulation implies that the standard deviations of the changes in the interest rates are proportional to the current levels of the rates.

The first term in each of these two equations represents the trend or deterministic part of the change in the interest rate. We anticipate that b_1 will exceed zero, so that when the long rate is (much) above the short rate, the short rate will be tending to rise. The trend part of the change in the long rate is derived by noting that the expected change in the long rate determines the expected rate of return on a consol bond, and that this should be related to the current levels of r and ℓ.

It is convenient to write Equations (1) and (2) in shorthand fashion as:

$$\tilde{\Delta}r = \mu_r + \sigma_r\tilde{\Delta}Z_r , \quad (1')$$

$$\tilde{\Delta}\ell = \mu_\ell + \sigma_\ell\tilde{\Delta}Z_\ell , \quad (2')$$

It is clear from the foregoing that there are two fundamental sources of uncertainty in the model—namely, the interest rates r and ℓ. Unanticipated changes in these two interest rates, or state variables, cause unanticipated changes in bond prices or returns. This is represented by writing the rate of return on a particular bond, j (j=1, . . . , n), as:

$$\tilde{R}_j = E[\tilde{R}_j] + \beta_j^r\tilde{\Delta}r' + \beta_j^\ell\tilde{\Delta}\ell' , \quad (3)$$

where β_j^r and β_j^ℓ measure the sensitivity of the bond's return to changes in r and ℓ, respectively, and $\tilde{\Delta}r'$ and $\tilde{\Delta}\ell'$ represent the unanticipated changes in the two interest rates.

The Equilibrium Condition

Consider a one-dollar portfolio formed by investing amounts X_1, X_2 and X_3 in bonds 1, 2 and 3 and investing the remaining $\$(1 - X_1 - X_2 - X_3)$ in the riskless, immediately maturing bond whose return is r. The return on this portfolio is:

$$\tilde{R}_p = r + X_1(\tilde{R}_1 - r) + X_2(\tilde{R}_2 - r) + X_3(\tilde{R}_3 - r). \quad (4)$$

Using Equation (3), the return on the portfolio can be expressed as:

$$\begin{aligned}\tilde{R}_p = r &+ X_1[E(\tilde{R}_1) - r] + X_2[E(\tilde{R}_2) - r] \\ &+ X_3[E(\tilde{R}_3) - r] \\ &+ (X_1\beta_1^r + X_2\beta_2^r + X_3\beta_3^r)\tilde{\Delta}r' \\ &+ (X_1\beta_1^\ell + X_2\beta_2^\ell + X_3\beta_3^\ell)\tilde{\Delta}\ell' . \quad (5)\end{aligned}$$

It is clear from Equation (5) that the portfolio proportions X_1, X_2 and X_3 may be chosen so that the coefficients of $\tilde{\Delta}r'$ and $\tilde{\Delta}\ell'$ are zero. The portfolio return is then riskless. The assumption of equilibrium then implies that the return on the portfolio equals r. Therefore, from Equation (5) it must be the case that if X_1, X_2 and X_3 are chosen to make the coefficients of $\tilde{\Delta}r$ and $\tilde{\Delta}\ell$ equal to zero, X_1, X_2 and X_3 must satisfy:

$$\begin{aligned}&X_1[E(\tilde{R}_1) - r] + X_2[E(\tilde{R}_2) - r] \\ &+ X_3[E(\tilde{R}_3) - r] = 0 . \quad (6)\end{aligned}$$

Equation (6) will be satisfied only if there is a relation between the bond risk premiums $[E(\tilde{R}_j) - r]$ and the bond return sensitivities, β_j^r and β_j^ℓ. This relation is the equilibrium condition:

$$E(\tilde{R}_j) - r = \lambda_1\beta_j^r + \lambda_2\beta_j^\ell , \quad (7)$$

where λ_1 and λ_2 are the same for all bonds and represent the market prices of short and long-term interest rate risk, respectively. The equilibrium condition may be expressed in words as:

Risk Premium = (λ_1 × Sensitivity to r)

+ (λ_2 × Sensitivity to ℓ) .

The condition is similar to the Capital Asset Pricing Model in that the risk premium is proportional to the beta coefficient.[3] It is in fact identical to the Arbitrage Pricing Theory when there are just two underlying factors.

The Discount Function

If we let $B(r,\ell,\tau)$ denote the market value of a particular bond with coupon rate c and time to maturity τ, then sensitivities β^r and β^ℓ can be expressed in terms of the partial derivatives of B(•) as:

$$\beta^r = \frac{B_r}{B},$$

$$\beta^\ell = \frac{B_\ell}{B},$$

so that Equation (7) becomes:

$$E(\tilde{R}_j) - r = \lambda_1 \frac{B_r}{B} + \lambda_2 \frac{B_\ell}{B}. \quad (8)$$

Moreover, the expected return on the bond depends on the expected changes in r and ℓ:

$$E(\tilde{R}_j) = \frac{1}{B}(-B_\tau + B_r\mu_r + B_\ell\mu_\ell + c + \ldots). \quad (9)$$

If we substitute for $E(\tilde{R}_j)$ in Equation (8), the result is a differential equation for the bond price B(•); and if we set the coupon rate, c, equal to zero and $B(r,\ell,0)$ equal to one, so that the bond value at maturity is unity, then the solution to the equation is $P(r,\ell,\tau)$—the value of a unit discount bond with τ periods to maturity, or the discount function.

The discount function depends on the parameters of the stochastic process—μ_r, σ_r, σ_ℓ and ρ—and also on the market price of short-term rate risk, λ_1.[4] We proceed by estimating the parameters of the stochastic process; given these, λ_1 is the value that best predicts the observed bond prices.

Model Estimation

For the purposes of estimation, we expressed the stochastic process for the two interest rates as:

$$\Delta r_t = a_1 + b_1(\ell_{t-1} - r_{t-1}) + r_{t-1}\epsilon_{1t},$$

$$\Delta \ell_t = \ell_{t-1}(a_2 + b_2 r_{t-1} + c_2 \ell_{t-1}) + \ell_{t-1}\epsilon_{1t}, \quad (10)$$

and estimated the coefficients using monthly data on interest rates and bond prices taken from the CRSP Government Bond File for the period December 1958 to December 1979. We took r as the yield on a 30-day Treasury bill and ℓ as the yield on a bond whose maturity exceeded 20 years (taking care to avoid flower bonds).

The results, presented in Table II, indicate a welcome stability in the parameter estimates between the two halves of the sample period. The coefficients of the short rate equation and the correlation coefficient are exceptions. It remains to be seen how important these instabilities are from the viewpoint of bond pricing. Meanwhile we note that, as hypothesized, b_1 exceeds zero, and the estimate is statistically significant for the entire period.

Estimating the Market Price of Risk

We estimated λ_1, the market price of risk parameter, by finding the value that yielded the "best" discount function, given the stochastic process parameters; the best discount function was that yielding bond prices closest to the observed prices. To avoid giving undue weight to individual bond maturities, we assigned all bonds with maturities up to 10 years to one of 10 equally weighted portfolios according to maturity; the values of these portfolios were predicted each month.

It must be noted that, since the discount function depends on both λ_1 and the stochastic process parameters, the optimal value of λ_1 will de-

Table II Estimated Parameters of the Stochastic Process (*standard errors in parentheses*)

$$\tilde{\Delta} r = [a_1 + b_1(\ell - r)] + rS_r\tilde{\Delta}Z_r$$

$$\tilde{\Delta}\ell = \ell(a_2 + b_2 r + c_2\ell) + \ell S_\ell \tilde{\Delta}Z_\ell$$

	a_1	b_1	a_2	b_2	c_2	S_r	S_ℓ	ρ
December 1958 to December 1979	−0.0887 (0.0526)	0.1102 (0.0301)	0.0089 (0.0069)	0.0036 (0.0017)	−0.0037 (0.0020)	0.1133	0.0298	0.2063
December 1958 to June 1969	−0.1809 (0.0754)	0.1882 (0.0480)	0.0151 (0.0200)	0.0047 (0.0037)	−0.0062 (0.0067)	0.1286	0.0233	0.0519
June 1969 to December 1979	−0.0135 (0.0826)	0.0377 (0.0369)	0.0319 (0.0221)	0.0044 (0.0023)	−0.0074 (0.0039)	0.0914	0.0349	0.3923

pend on the particular values assumed for the stochastic process parameters. This explains the radically different estimate of λ_1 reported in Table III using stochastic process parameter estimates derived from the first half of the sample period, rather than the whole period. It is to be expected that errors in the stochastic process parameter estimates and λ_1 will to some extent cancel out; whether this self-canceling of errors is sufficient to make the model usable outside the period over which it was estimated will be seen below.

Table III Estimates of λ_1
(standard errors in parentheses)

Bond Price Prediction Period	*Stochastic Process Estimation Period*	*Estimate of* λ_1
December 1958 to December 1979	December 1958 to December 1979	−0.450 (0.028)
December 1958 to June 1969	December 1958 to June 1969	−1.185 (0.019)
July 1969 to December 1979	December 1958 to December 1979	−0.283 (0.031)

Model Performance

We evaluated the performance of the model using both within-sample parameter estimates and out-of-sample estimates. For the within-sample estimates, data for the whole period December 1958 to December 1979 were used in estimating both the parameters of the stochastic process and λ_1. The out-of-sample estimates were obtained using data from the first half of the sample period—December 1958 to June 1969—and then evaluating the ability of the resulting model to predict bond prices over the second half of the sample period.

Table IV reports the price and yield prediction errors using within-sample parameter estimates for the whole period December 1958 to December 1979. The first line of the table presents the results of the predictions for each month of the sample period for all bonds with maturities up to 20 years (except flower bonds). The balance of the table shows the results obtained for predictions in December of each year. The root mean square error (RMSE) is a statistic that provides a measure of the average absolute error (unlike the mean error, which allows positive and negative errors to be offsetting). The root mean square price prediction error for the whole period was $1.58 per $100 of par value. The corresponding root mean square yield prediction error was 0.59 per cent. Too much attention should not be given to the yield predictions both because the model parameters were selected to minimize price prediction errors and because a small price prediction error may translate into a large yield error for short maturities.

The results in Table V are comparable to those in Table IV, except that they relate to the last half of the sample period; here the predictions are based on parameter estimates from the first half of the sample period, hence are true out-of-sample predictions. As one would expect, the out-of-sample prediction errors are somewhat greater—roughly twice as great as those of the within-sample predictions. It should be noted that we are using parameter estimates from one decade to predict over the following decade; this is a severe test, and no doubt the out-of-sample predictions could be improved by updating the parameter estimates.

Pricing Errors and Bond Returns

The price prediction errors may be due entirely to the deficiencies of the valuation model. On the other hand, they may be due to market inefficiencies, which would imply the existence of profit opportunities, or even to deficiencies in the price quotations, which would imply apparent profit opportunities. To discriminate between these alternatives, we treated the pricing errors as manifestations of market inefficiency and tested whether they were related in any systematic fashion to subsequent bond returns.

The issue of market efficiency is a controversial one, and we do not wish to be dogmatic about the implications of our findings. Within the context of our equilibrium model, however, our findings do imply market inefficiency. This is not to say that there do not exist other valuation models that would account for this anomaly, or even that the phenomenon is not due entirely to the quality of the price quotations.

The rate of return on a bond j was shown above to be given by the expression:

$$\tilde{R}_j = E[\tilde{R}_j] + \beta_j^r \tilde{\Delta} r' + \beta_j^\ell \tilde{\Delta}\ell' , \qquad (3)$$

where

$$\tilde{\Delta} r' = \tilde{\Delta} r - E[\tilde{\Delta} r] ,$$

$$\tilde{\Delta}\ell' = \tilde{\Delta}\ell - E[\Delta\ell'] .$$

Combining this with the equilibrium condition results in:

$$E[\tilde{R}_j] - r = \lambda_1 \beta_j^r + \lambda_2 \beta_j^\ell , \qquad (7)$$

Table IV Bond Price and Yield Predictions Using Within-Sample Estimates (December 1958—December 1979)[a]

	Number of Observations	Prices[b] Mean Error[c] ($)	Prices[b] RMSE ($)	Yields Mean Error[c] (%)	Yields RMSE (%)
Full Period	11,669	0.18	1.58	0.08	0.59
1958[d]	27	−0.03	1.29	0.32	0.38
59	33	−0.50	1.74	0.67	0.71
60	35	−0.19	0.65	0.04	0.18
61	38	−0.42	0.63	0.11	0.23
62	40	0.02	0.75	−0.15	0.33
63	40	−0.11	0.60	0.01	0.21
64	38	−0.23	0.71	0.12	0.28
65	35	−0.81	1.22	0.43	0.52
66	33	−0.44	0.77	0.25	0.35
67	34	−0.91	1.24	0.55	0.71
68	33	0.18	1.26	0.08	0.63
69	36	−2.21	2.50	1.39	1.55
70	40	0.27	0.43	−0.27	0.83
71	42	−0.01	0.52	−0.14	0.72
72	43	−1.12	1.48	0.38	0.54
73	45	1.70	2.40	−0.38	0.73
74	51	2.01	3.23	−0.27	1.10
75	60	0.94	1.54	−0.23	0.32
76	71	1.89	2.37	−0.66	0.72
77	81	0.68	1.51	0.01	0.39
78	87	0.20	1.59	0.48	0.94
79	94	0.17	1.91	0.67	1.31

a. Parameter estimates from whole sample period.
b. Per $100 par value.
c. Actual - predicted.
d. December.

Table V Bond Price and Yield Predictions Using Out-of-Sample Estimates (July 1969—December 1979)[a]

	Number of Observations	Prices[b] Mean Error[c] ($)	Prices[b] RMSE ($)	Yields Mean Error[c] (%)	Yields RMSE (%)
Full Period	7,045	2.53	3.90	−0.71	1.07
1969[d]	36	−1.09	1.31	0.92	1.21
70	40	1.65	2.07	−0.87	1.10
71	42	1.48	1.89	−0.70	0.93
72	43	0.19	0.52	−0.07	0.36
73	45	3.28	4.43	−0.89	1.16
74	51	4.02	5.65	−1.01	1.58
75	60	3.18	4.24	−1.10	1.16
76	71	4.01	4.96	−1.44	1.49
77	81	2.87	4.06	−0.82	0.99
78	87	2.13	3.83	−0.20	1.00
79	94	2.55	4.52	−0.18	1.33

a. Model estimated using only data from December 1958 to June 1969.
b. Per $100 par value.
c. Actual - predicted.
d. December.

which implies that the rate of return on the bond may be written:

$$\tilde{R}_j - r = (\lambda_1 + \tilde{\Delta}r')\beta_j^r + (\lambda_2 + \Delta\ell')\beta_j^\ell \ . \quad (11)$$

On the basis of Equation (11), the regression model estimated each period was:

$$R_{jt} - r_t = a_{ot} + a_{1t}\beta_{jt}^r + a_{2t}\beta_{jt}^\ell + a_{3t}E_{jt} + e_{jt}, (12)$$

where t denotes calendar time and E_{jt} is the pricing error for bond j at the beginning of month t. Market efficiency would imply that these errors would be unrelated to subsequent returns, so that a_{3t} would equal zero. However, to the extent that the error is attributable to temporary market disequilibria or deficiencies in the data, we should expect the coefficient to be positive. We constructed significance tests by comparing the means of the time series of $a_{ot} \ldots a_{3t}$ with

Table VI Bond Returns and Price Prediction Errors Using Valuation Model Estimated over December 1958 to December 1979 *(all taxable bonds with maturities less than 20 years; t-ratios in parentheses)*

	Holding Period = One Month						Holding Period = Three Months					
	Intercept ($\times 10^{-3}$)	β^r_{jt}	β^ℓ_{jt}	$COUP_{jt}$ ($\times 10^{-3}$)	MAT_{jt} ($\times 10^{-3}$)	E_{jt}	Intercept ($\times 10^{-3}$)	β^r_{jt}	β^ℓ_{jt}	$COUP_{jt}$ ($\times 10^{-3}$)	MAT_{jt} ($\times 10^{-3}$)	E_{jt}
December	0.485	-0.0233	0.0201				0.814	-0.0585	0.0638			
1958 to	(2.53)	(-0.56)	(1.11)					(1.09)	(-0.44)	(1.18)		
December	-1.22	-0.114	0.0352			0.136	-1.44	-0.139	0.0675			0.274
1979	(-3.72)	(-2.41)	(1.60)			(7.88)	(-1.79)	(-1.08)	(1.16)			(7.74)
	-0.817	-0.145	-0.0448	-0.104	-0.0304	0.146	-0.933	-0.129	-0.0519	-0.0915	-0.0595	0.311
	(-2.14)	(-2.08)	(-0.35)	(-2.02)	(-0.61)	(9.98)	(-1.02)	(-0.81)	(-0.18)	(-0.79)	(-0.53)	(8.93)
December	0.385	-0.0328	0.0197				0.147	-0.125	0.106			
1958 to	(1.99)	(-0.51)	(0.84)				(0.20)	(-0.67)	(1.43)			
June 1969	-0.0370	-0.0489	0.0536			0.0646	-0.440	-0.150	0.184			0.146
	(-0.14)	(-0.74)	(2.22)			(3.69)	(-0.62)	(-0.86)	(2.54)			(3.75)
	0.175	-0.140	-0.124	-0.184	-0.0812	0.105	0.453	-0.179	-0.0579	-0.322	-0.125	0.243
	(0.68)	(-1.14)	(-0.54)	(-3.11)	(-0.95)	(6.05)	(0.64)	(-0.70)	(-0.11)	(-2.39)	(-0.65)	(5.64)
July	0.586	-0.0136	0.0206				1.47	-0.0193	0.0626			
1969 to	(1.76)	(-0.27)	(0.73)				(1.12)	(-0.13)	(0.78)			
December	-2.43	-0.180	0.0166			0.209	-2.11	-0.125	0.0117			0.386
1979	(-4.11)	(-2.65)	(0.45)			(7.30)	(-1.47)	(-0.79)	(0.12)			(5.80)
	-1.83	-0.150	0.0359	-0.0229	0.0211	0.188	-0.799	-0.108	0.00243	-0.0931	0.0105	0.310
	(-2.56)	(-2.25)	(0.33)	(-0.27)	(0.42)	(8.13)	(-0.41)	(-0.69)	(0.01)	(-0.42)	(0.10)	(6.21)

Table VII Bond Returns and Price Prediction Errors Using Valuation Model Estimated over December 1958 to June 1969 *(all taxable bonds with maturities less than 20 years; t-ratios in parentheses)*

	Holding Period = One Month						Holding Period = Three Months					
	Intercept ($\times 10^{-3}$)	β^r_{jt}	β^ℓ_{jt}	$COUP_{jt}$ ($\times 10^{-3}$)	MAT_{jt} ($\times 10^{-3}$)	E_{jt}	Intercept ($\times 10^{-3}$)	β^r_{jt}	β^ℓ_{jt}	$COUP_{jt}$ ($\times 10^{-3}$)	MAT_{jt} ($\times 10^{-3}$)	E_{jt}
December	0.441	-0.0346	0.0220				0.757	-0.0958	0.0704			
1958 to	(1.97)	(-0.67)	(1.04)				(0.78)	(-0.53)	(1.07)			
December	-2.40	-0.312	-0.0863			0.152	-3.51	-0.521	-0.145			0.283
1979	(-5.73)	(-4.82)	(-2.77)			(8.25)	(-3.26)	(-2.75)	(-1.74)			(7.81)
	-1.95	-0.387	-0.320	-0.113	-0.767	0.173	-2.83	-0.565	-0.600	-0.0619	-0.159	0.341
	(-4.31)	(-5.11)	(-2.40)	(-2.17)	(-1.70)	(10.87)	(-2.52)	(-2.89)	(-1.80)	(-0.53)	(-1.49)	(9.29)
December	0.385	-0.0457	0.0218				-0.0453	-0.203	0.117			
1958 to	(1.69)	(-0.56)	(0.81)				(-0.05)	(-0.76)	(1.32)			
June 1969	-0.336	-0.157	0.0206			0.0657	-1.35	-0.440	0.119			0.146
	(-0.98)	(-1.69)	(0.72)			(3.46)	(-1.31)	(-1.62)	(1.31)			(3.37)
	-0.350	-0.364	-0.363	-0.200	-0.128	0.122	-0.804	-0.657	-0.549	-0.286	-0.226	0.258
	(-1.11)	(-2.91)	(-1.53)	(-3.28)	(-1.64)	(6.39)	(-0.90)	(-2.16)	(-0.95)	(-2.11)	(-1.24)	(5.61)
July	0.499	-0.0234	0.0221				1.37	-0.0387	0.0693			
1969 to	(1.28)	(-0.37)	(0.68)				(0.82)	(-0.19)	(0.73)			
December	-4.50	-0.469	-0.195			0.239	-5.45	-0.562	-0.368			0.436
1979	(-6.21)	(-5.30)	(-3.62)			(8.04)	(-2.85)	(-2.47)	(-2.84)			(6.39)
	-3.58	-0.412	-0.277	-0.0234	0.0250	0.225	-3.19	0.462	0.450	-0.0977	-0.0453	0.361
	(-4.30)	(-4.80)	(-2.32)	(-0.28)	(-0.56)	(9.07)	(-1.32)	(-2.02)	(-1.65)	(-0.43)	(-0.48)	(6.81)

their variances.

In Table VI, the valuation error E_{jt} was constructed from parameter estimates obtained using data from the whole sample period, and the regressions, using Equation (12), were estimated using data from the whole sample period for all taxable bonds with maturities up to 20 years. To control for possible misspecification of the valuation model, we included the coupon rate ($COUP_{jt}$) and maturity (MAT_{jt}) of the individual bonds as independent variables in some of the regressions. The left half of Table VI reports the results obtained when bond returns were calculated on a monthly basis; the right half of the table reports the results obtained using quarterly returns. In either case, the independent variables, including the price prediction error, were calculated as of the beginning of the observation interval.

The results reported in Table VI reveal a highly significant relation between the valuation error and the rate of return over the next time interval: Inspection of the coefficient of the price prediction error E_{jt} reveals that approximately 15 per cent of the error is corrected during the next month and 30 per cent over the next quarter. These results are insensitive to the inclusion of coupon and maturity variables, suggesting that they are not explicable in terms of model misspecification.

However, since the price prediction error used in the regressions reported in Table VI was derived from a valuation model whose parameters were estimated over the whole sample period, the prediction errors are not true *ex ante* forecasts, and it is possible that the results are attributable to testing the model on the data used to estimate it. To investigate this issue, the preceding analysis was repeated with the difference that the price prediction error was computed using valuation model parameters estimated over the first half of the sample period, December 1958 to June 1969. These results are reported in Table VII, which follows the same format as Table VI. It may be seen from the lower panel of this table that the price prediction error continues to have a highly significant effect on the subsequent return, even outside the sample period over which the valuation model parameters were estimated. Moreover, the coefficients of the price prediction errors are very similar to those reported in the previous table.

It appears from these results that this model may be used to predict the returns on straight bonds. The fact that the pricing errors are more strongly associated with subsequent quarterly returns than with subsequent monthly returns indicates that the performance of the model cannot be attributed solely to the quality of the data. Apart from the ability of the model to detect underpriced and overpriced bonds, its ability to yield bond price predictions under alternative interest rate scenarios should prove useful in designing and assessing the risk of bond portfolios. Finally, the success of the model in pricing straight bonds augurs well for its ability to deal with special bond features such as call and retraction provisions, as well as traded options on fixed income securities. ■

Footnotes

1. These are discussed at some length in M.J. Brennan and E.S. Schwartz, "Duration, Bond Pricing and Portfolio Management," in G.O. Bierwag, G. Kaufman and A. Toevs, eds., *Innovations in Bond Portfolio Management: Duration Analysis and Immunization* (JAI Press, forthcoming).
2. A more complete and rigorous discussion of the model and the empirical tests is to be found in M.J. Brennan and E.S. Schwartz, "An Equilibrium Model of Bond Pricing and a Test of Market Efficiency," *Journal of Financial and Quantitative Analysis*, forthcoming.
3. Of course, in the Capital Asset Pricing Model there is only a single beta representing the sensitivity to the market return.
4. λ_2 can be expressed in terms of μ_ℓ, σ_ℓ and r and ℓ; see M.J. Brennan and E.S. Schwartz, "A Continuous Time Approach to the Pricing of Bonds," *Journal of Banking and Finance* 3 (July), pp. 133-155.