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Michael J. Brennan and Eduardo S. Schwartz*

Abstract

This note explores the properties of some stock markets indices that are claimed to approximate a continuously rebalanced equally weighted portfolio.

I. Introduction

It is widely held that a geometric mean index of stock prices appreciates at the same rate as the value of a continuously rebalanced, equally weighted portfolio of the same stocks.[1] In this note we show that, while this proposition is correct if there is no uncertainty about the rate of increase in the individual stock prices, it is false if the stock returns are stochastic: the geometric mean index will grow more slowly than the value of the continuous equal-weight portfolio by a readily calculable amount that depends only on the covariance matrix of the instantaneous stock returns. We offer a rough estimate of the magnitude of the bias of the geometric mean index and calculate the equilibrium fee payable to a portfolio manager whose performance matches that of the geometric mean index.

Because in reality trade in stocks occurs only at discrete, stochastic intervals, the ideal of a continuously rebalanced portfolio is unattainable. In Section III, we consider an index that is intended to approximate the performance of a continuously rebalanced portfolio by recomputing the index value after each individual stock trade. Unfortunately, this index is asymptotically equivalent to the geometric mean index and is, therefore, biased.

II. The Bias of the Geometric Mean Index

To compare the behavior of the geometric mean index with that of the value of an ideal continuously rebalanced porfolio it is necessary to assume that the stocks trade continuously. We assume in addition that stock prices follow contin-

* Both authors, University of British Columbia, Vancouver, B.C., Canada, V6T 1Y8.

[1] Latane, Tuttle, and Young [4], Lorie and Hamilton [5], Rothstein [7]. Only Rothstein offers a proof of the properties of the geometric mean index and this proof implicitly neglects uncertainty.

uous sample paths so that the price dynamics are described by the general diffusion process

$$\frac{dP_i}{P_i} = \mu_i dt + \sigma_i dz_i, \quad i = 1,\ldots,n \tag{1}$$

where P_i is the price of stock i, t is calendar time, and dz_i is the increment of an Ito process with $E[dz_i] = 0$ and $dz_i dz_j = \rho_{ij} dt$.

Let E represent the value of a portfolio of n stocks that is continuously rebalanced to maintain an equal investment in each of the stocks. The instantaneous change in the value of this portfolio, neglecting dividends, is

$$\frac{dE}{E} = \frac{1}{n}\sum_{i=1}^{n}\mu_i dt + \frac{1}{n}\sum_{i=1}^{n}\sigma_i dz_i \,. \tag{2}$$

Let $G(P_{1t}, P_{2t}, \ldots, P_{nt})$ denote the geometric mean of the price-relatives of the same n stocks at time t

$$G \equiv \left[\frac{P_{1t}}{P_{10}} \cdot \frac{P_{2t}}{P_{20}} \cdots \frac{P_{nt}}{P_{n0}}\right]^{1/n} \,. \tag{3}$$

Using Ito's Lemma it is readily shown that

$$\begin{aligned} \frac{dG}{G} &= \left(\frac{1}{n}\sum_{i=1}^{n}\mu_i - B\right)dt + \frac{1}{n}\sum_{i=1}^{n}\sigma_i dz_i \\ &= \frac{dE}{E} - Bdt \end{aligned} \tag{4}$$

where
$$B \equiv \frac{1}{2}\sum_{i=1}^{n}\left(\frac{n-1}{n^2}\right)\sigma_i^2 - \frac{1}{2n^2}\sum\sum_{i \neq j}\rho_{ij}\sigma_i\sigma_j \,.$$

If stock prices are uncertain, then the bias factor $B > 0$, for $\sigma_i^2 + \sigma_j^2 > 2\sigma_i\sigma_j$. Thus, it is only when there is no uncertainty that the appreciation of the geometric mean index will match that of the continuously rebalanced portfolio. The negative bias in the geometric mean index also may be written as

$$B = \frac{1}{2}\left(\frac{n-1}{n}\right)\left(\bar{\sigma}^2 - \bar{\gamma}\right) \tag{5}$$

where
$$\bar{\sigma}^2 \equiv (1/n)\sum_{i=1}^{n}\sigma_i^2$$

$$\bar{\gamma} \equiv \left(\sum\sum_{i \neq j}\rho_{ij}\sigma_i\sigma_j\right) / \left(n^2 - n\right) \quad \text{and}$$

$\bar{\sigma}$ and $\bar{\gamma}$ are, respectively, the average variance and average covariance of the stocks in the sample.[2]

We may gain some idea of the order of magnitude of the bias by reference to Fisher and Lori's [3] study of the variability of common stock returns. They report that the average return on stocks was 9 percent, the average standard deviation of one-year price-relatives for one-stock portfolios was 40 percent and for a fully diversified portfolio was 22 percent. Assuming that the turns are lognormal, and treating the average variance as the square of the average standard deviation, we may infer that the average variance of instantaneous rates of return for single stock portfolios is 12.6 percent, and the corresponding variance of a fully diversified portfolio is 4.0 percent.[3] Taking the variance of a well diversified portfolio as an estimate of the average covariance, we arrive at an estimate of the bias in the geometric index for large n as B = æ (12.6 − 4.0) = 4.3 percent per year.[4]

Cootner [2] has suggested that an investment advisor should yearn to be compared with a geometric mean index. We now are able to quantify how much he should be willing to pay to satisfy this yearning. Comparing expressions (2) and (4) we see that a portfolio manager can match the geometric mean index performance by holding a continuously rebalanced portfolio and withdrawing cash for his own use at the continuous fractional rate B. This leaves the investor at the end of one year with a fraction $(1-\exp(-B))$ of the portfolio. Therefore, the manager would be willing to pay $\exp(-B)$ per dollar invested to manage the portfolio for one year if he were required only to match the performance of the geometric mean index. For B = 4.3 percent, this is 4.2 percent.

III. An Alternative Index of Portfolio Value

It has been proposed that the performance of a continously rebalanced portfolio can be closely approximated by an index that is incremented after each trade by a fraction (1/n) of the relative change in the price of the stock since its last trade. Indeed such an index is in use by at least one North American stock exchange. The value of this index at time t, $B(t)$, is given by

(6) $$\frac{B(t)}{B(0)} = \prod_{i=1}^{n} \prod_{j=1}^{m_i(t)} \left(1 + \frac{1}{n} r_{ij}\right) \quad \text{where}$$

[2] A few months after this note was written we became aware that a similar result was obtained by Modest and Sundaresan [6] in the context of stock index futures.

[3] Under the lognormal assumption, the variance rate of the instantaneous return (σ^2) is related to the mean (m) and variance (s^2) of the untransformed price relatives by $\sigma^2 = \ln(1 + s^2/(1 + m)^2)$. Note that the average variance does not equal the average standard deviation squared.

[4] Cootner [2] suggests that the Value Line Index, which is a geometric mean index, would appreciate about 3 percent per year faster if it were an arithmetic index.

r_{ij} is the return on stock i between trade j and the previous one; $m_i(t)$ is the total number of trades in stock i up to time t. Expression (6) also may be written as

$$\frac{B(t)}{B(0)} = \prod_{i=1}^{n}\left[\prod_{j=1}^{m_i(t)}\left(1 + \frac{1}{n}r_{ij}\right)^{n}\right]^{1/n} .$$

Then, using the properties of the exponential function, it follows that when the return between trades is small ($r_{ij} \rightarrow 0$)

$$\lim_{r_{ij}\rightarrow 0} \frac{B(t)}{B(0)} = \prod_{i=1}^{n}\left[\exp\left[\sum_{j=1}^{m_i(t)} r_{ij}\right]\right]^{1/n} = \prod_{i=1}^{n}\left(\frac{P_{it}}{P_{i0}}\right)^{1/n} \equiv G .$$

Thus, if the number of stocks in the index is large and the return between trades is small, the behavior of the index will follow closely that of the geometric mean index which, as we have seen, falls short of the theoretical ideal of the continuously rebalanced portfolio.

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