HEDGING INTEREST RATE RISK WITH OPTIONS ON AVERAGE INTEREST RATES

FRANCIS A. LONGSTAFF

Hedging interest rate risk has become one of the most common and important types of a financial manager's risk management activities. A classic example is for a firm to hedge its cost of funds by using an interest rate cap to place an upper bound on its borrowing costs. The hedge typically consists of a sequence of individual call options on the interest rate, with option expiration dates options coinciding with the borrower's interest payment dates. By purchasing an interest rate cap, the borrower can insure that the net interest cost for each individual payment is less than or equal to the cap rate.

Some firms may find it optimal to hedge the cost of individual interest payments. Most firms, however, would view their objective as hedging their *average* cost of funds during an accounting cycle, rather than hedging individual payments. For these firms, there are potential hedging vehicles that could prove far more cost-effective than a standard interest rate cap.

We describe one such hedging vehicle: a cap on the average interest rate during a period. Using a simple term structure model, we derive closed-form expressions for caps on average interest rates, illustrate their pricing and hedging properties, and contrast these properties with those of standard interest rate caps. We show that a cap on the average rate can cost far less than a conventional cap.

This is consistent with Merton [1973], who shows that an option on an average is worth less than a portfolio of options. What is different here, however, is that mean reversion and the slope of the term structure play an additional role in determining the relation between the two prices. We also show that caps on the average rate are generally less sensitive to changes in

FRANCIS A. LONGSTAFF is a professor at the Anderson Graduate School of Management of the University of California, Los Angeles.

interest rates and are correspondingly easier to hedge.

I. PRICING CAPS ON AVERAGE RATES

We derive all our results within the context of the simplest version of the well-known Vasicek [1977] term structure framework. The basic nature of the results, however, would be the same in a more general and realistic type of term structure model.

We first derive the price of a standard interest rate cap in this framework and discuss its pricing behavior. We then derive the price of a cap on an average rate and contrast its properties with those of a standard cap.

Let r denote the short-term riskless interest rate. In the Vasicek [1977] model, the risk-adjusted dynamics for the riskless rate can be expressed as

$$dr = (\alpha - \beta r)dt + \sigma dZ \tag{1}$$

where α incorporates the market price of interest rate risk, and Z is a standard Wiener process. The price D(r, T) of a zero-coupon bond with maturity T is given by

$$D(r, T) = \exp(A(T) - B(T)r) \tag{2}$$

where

$$A(T) = \left(\frac{\eta^2}{2\beta^2} - \frac{\alpha}{\beta}\right)T + \left(\frac{\eta^2}{\beta^3} - \frac{\alpha}{\beta^2}\right)(\exp(-\beta T) - 1) - \left(\frac{\eta^2}{4\beta^3}\right)(\exp(-2\beta T) - 1),$$

$$B(T) = \frac{1 - \exp(-\beta T)}{\beta}$$

Let H(r, T) denote the value of a generic interest rate-sensitive contingent claim that makes a payment of F(r) at time T. In this framework, H(r, T) satisfies the partial differential equation

$$\frac{\sigma^2}{2}H_{rr} + (\alpha - \beta r)H_r - rH = H_T \tag{3}$$

subject to the expiration condition H(r, 0) = F(r). The results in Longstaff [1990] and Longstaff and Schwartz [1993], however, can be used to show that the value of this derivative security can always be represented in the form of a certainty equivalent

$$H(r, T) = D(r, T)\, E[F(r)] \tag{4}$$

where the expectation is taken with respect to the certainty-equivalent dynamics for the riskless rate

$$dr = [\alpha - \beta r - \sigma^2 B(T - t)]dt + \sigma dZ \tag{5}$$

where B(·) is as defined in (2).

Let C(r, K, T) be the value of a simple cap that pays max(0, r − K) at time T. A standard or full cap can be viewed as a portfolio of simple caps, where the expiration dates of the individual simple caps correspond to the periodic interest payment dates of the underlying debt instrument being hedged. Thus, the value of a full cap equals the sum of values of the individual simple caps. From (5), it can be shown that the the time T value of the riskless rate r_T implied by the certainty-equivalent dynamics is normally distributed with mean M(r, T) and variance V(T), where

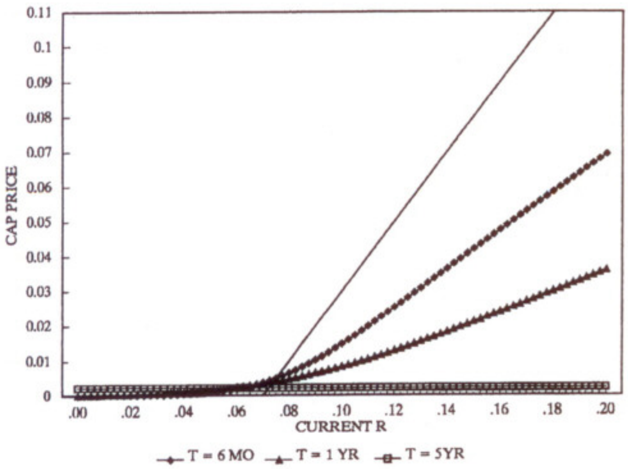
$$M(r, T) = re^{-\beta T} + \left(\frac{\alpha}{\beta} - \frac{\sigma^2}{\beta^2}\right)(1 - e^{-\beta T}) + \frac{\sigma^2}{2\beta^2}(1 - e^{-2\beta T}),$$

$$V(T) = \frac{\sigma^2}{2\beta}(1 - e^{-2\beta T}) \tag{6}$$

With these results, we can solve for the value of a simple interest rate cap by directly taking the expectation in (4):

$$C(r, K, T) = D(r, T)\sqrt{\frac{V(T)}{2\pi}} \times \exp\left(-\frac{(M(r, T) - K)^2}{2V(T)}\right) + D(r, T)(M(r, T) - K) \times$$

EXHIBIT 1 ■ Value of Call Option on Level of Riskless Rate r



Notes: The parameter values used are $\alpha = 0.06$, $\beta = 1$, and $\sigma = 0.025$. The strike price of the option is 0.07.

$$N\left(\frac{M(r, T) - K}{\sqrt{V(T)}}\right) \quad (7)$$

where $N(\cdot)$ is the cumulative normal distribution function.

Although a simple cap is a call option on the level of the riskless rate, the properties of cap prices are fundamentally different from those implied by the Black-Scholes model. The major reason for these differences is not that the Vasicek model implies a different distribution from the Black-Scholes [1973] model. Rather, the primary difference is that the interest rate is not the price of a traded asset, even though it is computed from market prices. Thus, the dynamics of the interest rate can be very different from those followed by a traded asset in a risk-neutral valuation framework.

To illustrate some of these differences, Exhibit 1 graphs the value of a cap as a function of the underlying riskless rate for various expiration dates. Note that the value of a simple cap can be less than its intrinsic value. The intuition for this is related to the mean reversion of interest rates. Although a cap may currently be in-the-money because rates are high, the reversion of rates back toward their long-run average value causes the effective moneyness of caps struck at rates above the long-run mean to decay over time. Note also that the price of a cap is not necessarily an increasing function of T.

A second important difference is that the delta of a cap, which is given by the derivative of C(r, K, T) with respect to r,

$$C_r(r, K, T) = -B(T)C(r, K, T) + e^{-\beta T} D(r, T) N\left(\frac{M(r, T) - K}{\sqrt{V(T)}}\right) \quad (8)$$

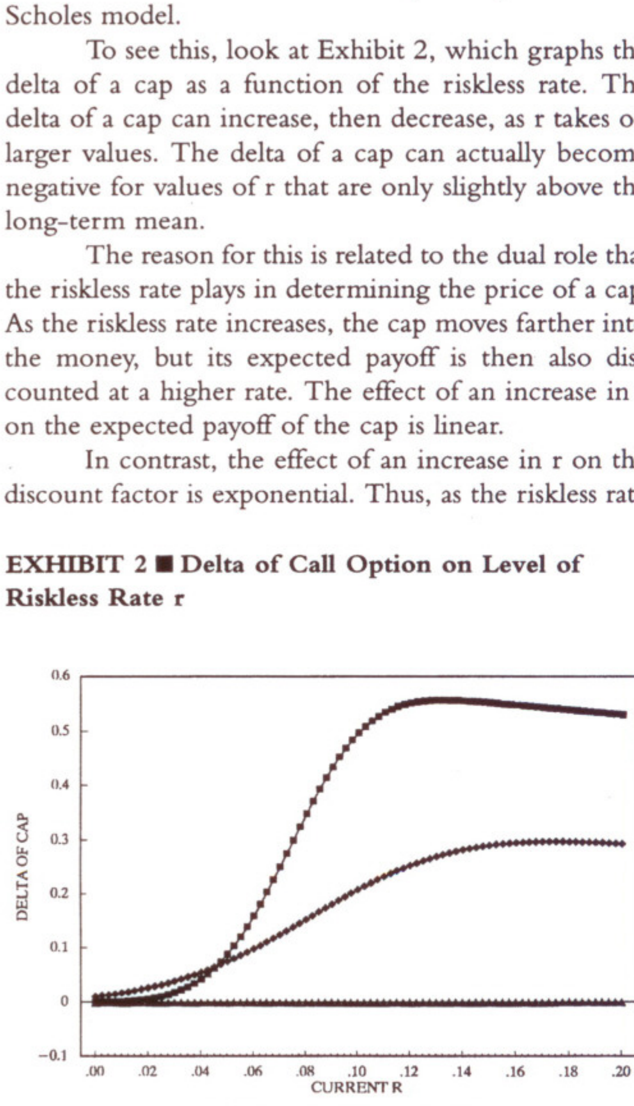
can be far less than the delta implied by the Black-Scholes model.

To see this, look at Exhibit 2, which graphs the delta of a cap as a function of the riskless rate. The delta of a cap can increase, then decrease, as r takes on larger values. The delta of a cap can actually become negative for values of r that are only slightly above the long-term mean.

The reason for this is related to the dual role that the riskless rate plays in determining the price of a cap. As the riskless rate increases, the cap moves farther into the money, but its expected payoff is then also discounted at a higher rate. The effect of an increase in r on the expected payoff of the cap is linear.

In contrast, the effect of an increase in r on the discount factor is exponential. Thus, as the riskless rate

EXHIBIT 2 ■ Delta of Call Option on Level of Riskless Rate r



Notes: The parameter values used are $\alpha = 0.06$, $\beta = 1$, and $\sigma = 0.025$. The strike price of the option is 0.07.

increases, the discount effect eventually offsets the payoff effect and the cap becomes a decreasing function of the riskless rate. The Black [1976] model, which practitioners often use to value and hedge caps, misses the discount effect and can actually give the wrong sign for the delta of in-the-money caps.

With these preliminaries, let us now consider the pricing of caps on the average interest rate. Let X_T denote the average interest rate between now and time T, where the average is computed as

$$X_T = \frac{1}{T}\int_0^T r_t dt \tag{9}$$

Solving the stochastic differential equation in (5) gives the expression for the riskless rate at any time t, where $0 \le t \le T$:

$$r_t = re^{-\beta t} + \left(\frac{\alpha}{\beta} - \frac{\sigma^2}{\beta^2}\right)(1 - e^{-\beta T}) + \frac{\sigma^2}{2\beta^2}e^{-\beta T}(e^{\beta t} - e^{-\beta t}) + \sigma e^{-\beta t} \times \int_0^t e^{\beta s} dZ \tag{10}$$

Substituting this expression into (9) and evaluating the integral implies that the average interest rate X_T is normally distributed with mean P(r, T) and variance Q(T), where

$$P(r, T) = \left(\frac{\alpha}{\beta} - \frac{\sigma^2}{\beta^2}\right) + \left(\frac{r}{T\beta} - \frac{\alpha}{T\beta^2} + \frac{\sigma^2}{T\beta^3}\right)(1 - e^{-\beta T}) + \frac{\sigma^2}{2T\beta^3}(1 - e^{-\beta T})^2,$$

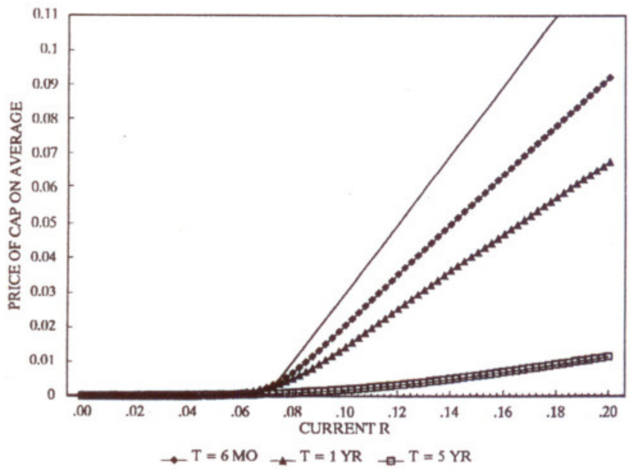
$$Q(T) = \frac{\sigma^2}{T\beta^2} - \frac{2\sigma^2}{T^2\beta^3}(1 - e^{-\beta T}) + \frac{\sigma^2}{2T^2\beta^3}(1 - e^{-2\beta T}) \tag{11}$$

Let CA(r, K, T) be the value of a cap on the average interest rate with payoff $\max(0, X_T - K)$ at time T. Substituting this payoff into (4) and evaluating the expectation directly leads to the expression for CA(r, K, T):

$$CA(r, K, T) = D(r, T)\sqrt{\frac{Q(T)}{2\pi}} \times \exp\left(-\frac{(P(r, T) - K)^2}{2Q(T)}\right) + D(r, T)(P(r, T) - K) N\left(\frac{(P(r, T) - K}{\sqrt{Q(T)}}\right) \tag{12}$$

Although a cap on the average interest rate, this contingent claim shares many of the same counterintuitive properties of a simple cap on the interest rates. For example, Exhibit 3 shows that the price of a cap on the average interest rate is not always an increasing function of T. In particular, the price of the cap on the average rate can either be an increasing or decreasing function of the time until expiration for smaller values of T, but ultimately becomes a decreasing function of T as the

EXHIBIT 3 ■ Value of Call Option on Average Riskless Rate r



Notes: The parameter values used are $\alpha = 0.06$, $\beta = 1$, and $\sigma = 0.025$. The strike price of the option is 0.07.

time to expiration increases. As $T \to \infty$, the price of a cap on the average rate converges to zero.

The hedging properties of a cap on the average rate are also quite different from those for options on traded assets. The delta for a cap on the average interest rate is given by the expression

$$CA_r(r, K, T) = -B(T)CA(r, K, T) + \frac{B(T)}{T}D(r, T)N\left(\frac{(P(r, T) - K}{\sqrt{Q(T)}}\right) \quad (13)$$

Exhibit 4 shows that the delta of these types of caps can also decrease as r increases. If fact, the delta of a cap on the average interest rate can actually become negative for sufficiently large values of r. This means that, in some situations, it may be possible to hedge a long position in a cap with another long position in a cap. This could make these types of derivative securities less risky from the perspective of a financial institution selling caps on the average rate since there will likely be some diversification among their positions.

Note that the delta of a cap on the average rate is typically bounded well below one. Hence, standard intuition about hedging derivative positions implied by the Black-Scholes model is inapplicable.

The expression in (12) is the value of a cap on the average rate when the calculation of the average begins at time zero. When the calculation begins prior to the current period, say, N periods in the past, this valuation expression must be modified slightly. This can be done by noting that the actual payoff function for the cap can be expressed as

$$\max\left(0, \frac{TA_T}{T + N} + \frac{NA_N}{T + N} - K\right) \quad (14)$$

where A_N is the average to date, which is known at the current time. This payoff can be rearranged as $T/(T + N) \max(0, A_T - \hat{K})$, where $\hat{K} = K(1 + N/T) - NA_N/T$. Thus, the value of a cap on the average, where the calculation of the average begins N periods previously, is given by $T/(T + N)CA(r, \hat{K}, T)$.

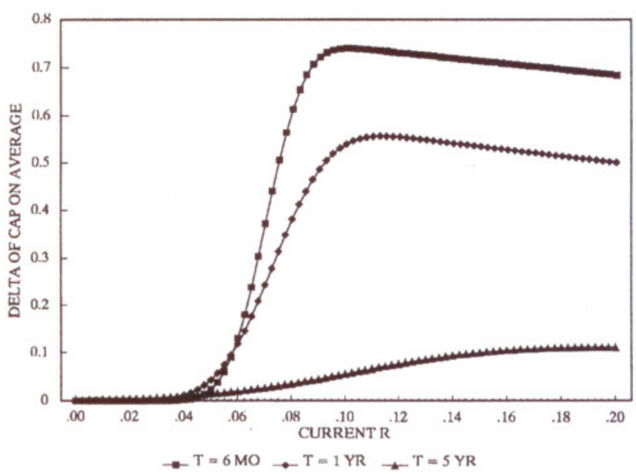
II. A COMPARISON OF HEDGING COSTS

Although both a cap and a cap on the average interest rate can be used to hedge against unexpected increases in the cost of financing, the two types of hedges are fundamentally different. To see this, consider a stylized hedging situation. A firm borrows $1 for a one-year period at a floating rate of r, where interest accrues and is paid daily. The firm could hedge each of its daily payments by purchasing a full cap of 365 individual simple caps. Alternatively, the firm could hedge its annual total cost of funds by purchasing a one-year cap on the average value of r.

To compare the costs of the two hedging strategies, Exhibit 5 presents the total cost of the full cap, and Exhibit 6 presents the cost of the corresponding cap on the average rate for various values of r and K. Exhibit 7 reports the difference between the costs of the hedging instruments, and Exhibit 8 reports the ratio of the cost of the cap on the average rate to the full cap. The numbers show that the cost of hedging the average financing costs of the firm is generally much lower than the costs of capping each individual payment.

For example, the cost savings from capping the average rather than individual interest payments can exceed 99% when the current rate is low and the cap on the average is struck at a higher value of K. The cost savings are lower when the current rate is above its long-term average value and the cap on the average is struck at a lower rate, but they are still on the order of 5%. In dollar terms, the cost savings vary significantly

EXHIBIT 4 ■ Delta of Call Option on Average Riskless Rate r



Notes: The parameter values used are $\alpha = 0.06$, $\beta = 1$, and $\sigma = 0.025$. The strike price of the option is 0.07.

EXHIBIT 5 ■ Total Cost of Full Cap on Daily Floating-Rate Interest Payments

Current Riskless Rate	Cap Struck at					
	K = 0.04	K = 0.05	K = 0.06	K = 0.07	K = 0.08	K = 0.09
r = 0.02	0.00389	0.00148	0.00045	0.00011	0.00002	0.00000
r = 0.03	0.00605	0.00249	0.00082	0.00021	0.00004	0.00001
r = 0.04	0.00940	0.00415	0.00149	0.00042	0.00009	0.00001
r = 0.05	0.01423	0.00695	0.00270	0.00083	0.00020	0.00004
r = 0.06	0.01980	0.01133	0.00494	0.00165	0.00043	0.00009
r = 0.07	0.02563	0.01659	0.00879	0.00338	0.00095	0.00021
r = 0.08	0.03155	0.02224	0.01367	0.00667	0.00223	0.00052
r = 0.09	0.03749	0.02806	0.01906	0.01109	0.00499	0.00143
r = 0.10	0.04342	0.03394	0.02471	0.01615	0.00892	0.00370
r = 0.11	0.04933	0.03984	0.03049	0.02157	0.01358	0.00715

Notes: Time horizon is one year. The parameters used are $\alpha = 0.06$, $\beta = 1$, and $\sigma = 0.025$. The underlying principal amount is \$1.

across different values of r and K. In general, however, the dollar cost savings are generally higher when the current interest rate is above its long-term average.

There are several reasons why the cost of a cap on the average rate is less than the cost of the full cap. The first follows from Merton [1973], who shows that an option on an average is worth less than the corresponding portfolio of options. This is because the valuation operator in (4) involves taking the expectation of a random payoff. The average of a convex payoff is always greater than or equal to the payoff function evaluated at the average.

A second reason is related to the fact that averages are less volatile than the underlying process being averaged. This is because the averaging process tends to allow some of the randomness to be diversified away. Thus, the volatility of an average is typically less than the average volatility of the random variables being averaged.

Finally, mean reversion plays an important role in explaining why a cap on the average rate costs less than a full cap. The mean reversion of the interest rate implies that the volatility per unit time of the interest rate process converges to zero as the length of the horizon increases. In contrast, the total volatility of returns in the Black-Scholes model is assumed to grow linearly with T, which implies that the volatility per unit time is constant for all horizons. This feature implies that the cost difference between calls on average rates and calls on the level

EXHIBIT 6 ■ Cost of Cap on Average Interest Rate

Current Riskless Rate	Cap Struck at					
	K = 0.04	K = 0.05	K = 0.06	K = 0.07	K = 0.08	K = 0.09
r = 0.02	0.00188	0.00029	0.00002	0.00000	0.00000	0.00000
r = 0.03	0.00439	0.00102	0.00012	0.00001	0.00000	0.00000
r = 0.04	0.00830	0.00273	0.00050	0.00005	0.00000	0.00000
r = 0.05	0.01328	0.00580	0.00157	0.00022	0.00002	0.00000
r = 0.06	0.01883	0.01018	0.00380	0.00083	0.00009	0.00000
r = 0.07	0.02455	0.01541	0.00742	0.00231	0.00040	0.00003
r = 0.08	0.03026	0.02101	0.01217	0.00510	0.00130	0.00017
r = 0.09	0.03591	0.02668	0.01755	0.00920	0.00327	0.00067
r = 0.10	0.04149	0.03231	0.02314	0.01423	0.00660	0.00195
r = 0.11	0.04700	0.03787	0.02875	0.01968	0.01111	0.00446

Notes: Time horizon is one year. The parameters used are $\alpha = 0.06$, $\beta = 1$, and $\sigma = 0.025$. The underlying principal amount is \$1.

EXHIBIT 7 ■ Difference Between Total Cost of Full Cap on Daily Floating-Rate Interest Payments and Cost of Cap on Average Interest Rate

Current Riskless Rate	Cap Struck at					
	K = 0.04	K = 0.05	K = 0.06	K = 0.07	K = 0.08	K = 0.09
r = 0.02	0.00200	0.00119	0.00043	0.00011	0.00002	0.00000
r = 0.03	0.00166	0.00147	0.00070	0.00021	0.00004	0.00001
r = 0.04	0.00110	0.00142	0.00099	0.00038	0.00009	0.00001
r = 0.05	0.00095	0.00115	0.00113	0.00061	0.00018	0.00004
r = 0.06	0.00097	0.00115	0.00114	0.00083	0.00034	0.00008
r = 0.07	0.00108	0.00119	0.00138	0.00106	0.00056	0.00017
r = 0.08	0.00129	0.00124	0.00150	0.00158	0.00093	0.00035
r = 0.09	0.00158	0.00138	0.00151	0.00190	0.00171	0.00076
r = 0.10	0.00193	0.00163	0.00157	0.00193	0.00232	0.00175
r = 0.11	0.00233	0.00197	0.00174	0.00189	0.00247	0.00270

Notes: Time horizon is one year. The parameters used are $\alpha = 0.06$, $\beta = 1$, and $\sigma = 0.025$. The underlying principal amount is $1.

of the rate is much greater than would be the case if the underlying were the price of a traded asset.

Finally, it is useful to compare the deltas of the two types of hedging instruments. Exhibit 9 presents the deltas of the full caps shown in Exhibit 5, and Exhibit 10 presents the deltas of the caps on the average rate shown in Exhibit 6. Note that the deltas for the caps on the average rate are generally smaller than the deltas for the caps. Thus, caps on the average rate can be both less expensive and less risky than the full cap.

It is important to note, however, that there are some exceptions. For example, even though the cap on the average is less expensive than the full cap for smaller values of r and K, the delta of the cap on the average is actually slightly higher.

In considering whether to hedge using a cap on the average or a full cap, a risk manager would need to assess the relative trade-offs between cost savings and the riskiness of the hedging vehicle, particularly when the term structure is steeply upward-sloping.

III. CONCLUSION

We have illustrated that caps on average interest

EXHIBIT 8 ■ Ratio of Cost of Cap on Average Interest Rate to Total Cost of Full Cap on Daily Floating-Rate Interest Payments

Current Riskless Rate	Cap Struck at					
	K = 0.04	K = 0.05	K = 0.06	K = 0.07	K = 0.08	K = 0.09
r = 0.02	0.48398	0.19499	0.04727	0.00657	0.00051	0.00002
r = 0.03	0.72576	0.40835	0.14634	0.03117	0.00378	0.00025
r = 0.04	0.88297	0.65748	0.33634	0.10732	0.02014	0.00214
r = 0.05	0.93323	0.83441	0.58161	0.27007	0.07689	0.01276
r = 0.06	0.95117	0.89813	0.76930	0.50004	0.21085	0.05372
r = 0.07	0.95798	0.92848	0.84352	0.68562	0.41509	0.15934
r = 0.08	0.95922	0.94438	0.89010	0.76388	0.58347	0.32963
r = 0.09	0.95791	0.95072	0.92070	0.82911	0.65679	0.46669
r = 0.10	0.95556	0.95188	0.93655	0.88079	0.74010	0.52647
r = 0.11	0.95280	0.95065	0.94282	0.91241	0.81793	0.62294

Notes: Time horizon is one year. The parameters used are $\alpha = 0.06$, $\beta = 1$, and $\sigma = 0.025$. The underlying principal amount is $1.

EXHIBIT 9 ■ Delta For Full Cap on Daily Floating-Rate Interest Payments

Current Riskless Rate	Cap Struck at					
	K = 0.04	K = 0.05	K = 0.06	K = 0.07	K = 0.08	K = 0.09
r = 0.02	0.17340	0.07769	0.02756	0.00750	0.00153	0.00023
r = 0.03	0.26672	0.12802	0.04927	0.01470	0.00332	0.00055
r = 0.04	0.41669	0.21310	0.08836	0.02873	0.00712	0.00132
r = 0.05	0.53253	0.36342	0.16140	0.05675	0.01536	0.00313
r = 0.06	0.57403	0.49481	0.30606	0.11558	0.03386	0.00753
r = 0.07	0.58913	0.55083	0.44892	0.24896	0.07830	0.01880
r = 0.08	0.59376	0.57554	0.51924	0.39767	0.19641	0.05041
r = 0.09	0.59385	0.58598	0.55505	0.48008	0.34502	0.15146
r = 0.10	0.59191	0.58948	0.57314	0.52709	0.43576	0.29494
r = 0.11	0.58902	0.58946	0.58165	0.55402	0.49253	0.38970

Notes: Time horizon is one year. The parameters used are $\alpha = 0.06$, $\beta = 1$, and $\sigma = 0.025$. The underlying principal amount is $1.

rates can provide an important alternative hedging vehicle for managing borrowing costs. This type of derivative security is always less costly than the corresponding full cap, and is generally less sensitive to changes in interest rates. By capping the average rate rather than each individual interest payment, the firm is adopting a longer-term perspective to hedging.

The results suggest, moreover, that there can be considerable cost savings to a firm using these instruments to cap its financing costs even if the horizon is only one year. Clearly, the savings could be substantially higher if a firm were willing to cap its average cost of funds over a two-year, three-year, or even longer horizon.

REFERENCES

Black, F. "The Pricing of Commodity Contracts." *Journal of Financial Economics*, 3 (1976), pp. 167-179.

Black, F., and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, 81 (1973), pp. 637-654.

Longstaff, F. "The Valuation of Options on Yields." *Journal of Financial Economics*, 26 (1990), pp. 97-121.

Longstaff, F., and E. Schwartz. "A Simple Approach to Valuing Risky Fixed and Floating Rate Debt and

EXHIBIT 10 ■ Delta of Cap on Average Interest Rate

Current Riskless Rate	Cap Struck at					
	K = 0.04	K = 0.05	K = 0.06	K = 0.07	K = 0.08	K = 0.09
r = 0.02	0.18232	0.04073	0.00407	0.00017	0.00000	0.00000
r = 0.03	0.32328	0.11401	0.01912	0.00140	0.00004	0.00000
r = 0.04	0.45376	0.23672	0.06442	0.00801	0.00042	0.00001
r = 0.05	0.53545	0.37824	0.15857	0.03272	0.00299	0.00011
r = 0.06	0.56795	0.48981	0.29355	0.09642	0.01488	0.00099
r = 0.07	0.57294	0.54841	0.42650	0.20981	0.05290	0.00604
r = 0.08	0.56826	0.56641	0.51544	0.34886	0.13686	0.02605
r = 0.09	0.56138	0.56585	0.55395	0.46564	0.26481	0.08089
r = 0.10	0.55421	0.55985	0.56177	0.53146	0.39890	0.18461
r = 0.11	0.54708	0.55283	0.55801	0.55406	0.49463	0.31985

Notes: Time horizon is one year. The parameters used are $\alpha = 0.06$, $\beta = 1$, and $\sigma = 0.025$. The underlying principal amount is $1.

Determining Swap Spreads." Working Paper, UCLA, 1993.

Merton, R.C. "The Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science*, 4 (1973), pp. 141-183.

Vasicek, O. "An Equilibrium Characterization of the Term Structure." *Journal of Financial Economics*, 5 (1977), pp. 177-188.