

Temporal Aggregation and the Continuous-Time Capital Asset Pricing Model

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ABSTRACT

We examine how the empirical implications of the Capital Asset Pricing Model (CAPM) are affected by the length of the period over which returns are measured. We show that the continuous-time CAPM becomes a multifactor model when the asset pricing relation is aggregated temporally. We use Hansen's Generalized Method of Moments (GMM) approach to test the continuous-time CAPM at an unconditional level using size portfolio returns. The results indicate that the continuous-time CAPM cannot be rejected. In contrast, the discrete-time CAPM is easily rejected by the tests. These results have a number of important implications for the interpretation of tests of the CAPM which have appeared in the literature.

FEW RESULTS IN FINANCE are as familiar or as widely used as the linear relation between expected returns and market betas implied by the Capital Asset Pricing Model (CAPM). However, because both the discrete- and continuous-time versions of the CAPM lead to this linear relation,¹ the literature often fails to distinguish between their respective empirical implications. For example, discrete- and continuous-time models are frequently used interchangeably to motivate empirical tests that use discretely observed return data.²

This paper examines how the empirical implications of the CAPM are affected by the length of the period over which returns are measured.³ We show that, if this period differs from the implicit time frame of the model, then the familiar linear CAPM relation need not hold for the observed returns. For example, the CAPM can imply a multifactor expression for expected returns when the asset pricing relation is properly aggregated. This aspect has many important implications for applied work as well as the interpretation of empirical tests of the CAPM that have appeared in the literature.

We illustrate the importance of the temporal aggregation issue by deriving the

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¹ The CAPM has been derived in a discrete-time setting by Sharpe (1964), Lintner (1965), Black (1972), Constantinides (1980), and many others. Merton (1971, 1973) and Cox, Ingersoll, and Ross (1985b) (implicitly) have derived the continuous-time CAPM relation.

² For example, see Chan, Chen, and Hsieh (1985), Chen, Roll, and Ross (1986), Gibbons and Ferson (1985), Ferson, Kandel, and Stambaugh (1987), and Ferson (1988).

³ Grossman, Melino, and Shiller (1987) discuss the effects of aggregation over time on the continuous-time CAPM. See also Christiano (1984) and Bergstrom (1984).

cross-sectional restriction on the means of discretely observed returns imposed by a simple version of the continuous-time CAPM. This formulation of the continuous-time CAPM is based on Cox, Ingersoll, and Ross (CIR (1985b)) and has the advantage of allowing both expected returns and covariances to be time varying.⁴ The cross-sectional restriction is obtained by aggregating the continuous-time price processes temporally, deriving closed-form expressions for the continuous-time parameters in terms of the moments of discretely observed returns, and then substituting these expressions into the continuous-time asset pricing relation.

The resulting form of the continuous-time CAPM has a number of interesting and important implications for the properties of observable returns. We show that the relative riskiness of assets (as measured by their market betas) can depend on the length of the period over which returns are computed. This means that, although asset A may appear riskier than asset B when daily returns are used to compute market betas, the opposite could be true for some other return measurement period.⁵ In addition, we show that the relation between expected returns and market betas need not be strictly linear. However, for returns measured over short intervals such as a month, the continuous-time CAPM implies that expected returns can be approximated as linear functions of unconditional market covariances, own variances, and first-order autocovariances. Thus, expected returns are approximated by a three-factor model in the continuous-time CAPM.

We propose a test of this linear moment restriction using the Generalized Method of Moments (GMM) technique of Hansen (1982). Because these tests are performed at an unconditional level, long time series of returns can be used in order to obtain precise estimates of the sample moments as in Chan and Chen (1988). The simple functional form of the moment equations allows us to separate the data from the parameters and compute the GMM estimators directly in a single-step procedure similar to that used by Gibbons and Ramaswamy (1986). The empirical results indicate that the continuous-time CAPM cannot be rejected even when confronted with returns from portfolios formed on the basis of size. In contrast, the empirical tests strongly reject the discrete-time (single-factor) CAPM. The results also suggest that unconditional variances and autocovariances have significant incremental explanatory power for cross-sectional differences in unconditional returns.

Section I discusses the continuous-time CAPM and derives its implications for the properties of returns measured over discrete intervals. Section II discusses the GMM methodology. Section III describes the data and the formation of portfolios. Section IV presents the empirical results. Section V summarizes the paper and makes concluding remarks.

⁴ Gibbons and Ferson (1985), Ferson, Kandel, and Stambaugh (1987), and Ferson (1988) examine models which allow expected returns to vary through time. Bollerslev, Engle, and Wooldridge (1988), Bodurtha and Mark (1987), Chan and Chen (1988), and Harvey (1988) study specifications which allow for time-varying covariances.

⁵ This is because market betas computed from discretely observed returns can be expressed as weighted combinations of the assets' continuous-time parameters. As the return measurement period varies, the weights change (nonproportionally) and the relative rankings of the betas can shift.

I. The Temporally Aggregated CAPM

In this section, we derive the restrictions imposed by the continuous-time CAPM on returns measured over discrete intervals of time. We focus on the continuous-time CAPM because it unambiguously identifies the period over which the asset pricing relation holds (an instant). This property is not shared by the discrete-time CAPM, which leaves the relevant portfolio holding period unspecified. We will show in this section that the implicit time frame is a major factor in determining the properties of expected returns in the CAPM.

In deriving these restrictions, we use the well-known continuous-time economy of CIR (1985b) as the basic setting for the model. This intuitively appealing framework has the advantage of providing a simple yet complete characterization of production and investment opportunities as well as technological change in a general equilibrium context. In addition, this framework has the realistic feature of allowing expected returns and market betas to vary through time. This feature is particularly important in view of the mounting empirical evidence of time variation in expected returns and risk measures.⁶

In the CIR (1985b) framework, there are a fixed number of identical individuals with time-additive logarithmic preferences who allocate their wealth between consumption and physical investment. All physical investment is performed by stochastic constant returns to scale technologies or firms which produce a single good. The returns from investing P_i in the i th firm are governed by the stochastic differential equation:

$$\frac{dP_i}{P_i} = \alpha_i X dt + \sigma_i \sqrt{X} dZ_i, \quad (1)$$

where α_i and σ_i are constants, Z_i is a scalar⁷ Wiener process, and X is a state variable which induces random technological change. The state variable X has the following dynamics:

$$dX = \kappa(\mu - X)dt + s\sqrt{X}dZ_X, \quad (2)$$

where κ , μ , and s are constants and Z_X is also a scalar process. Together, the values of each firm and the current level of the state variable completely describe the state of the economy and the distribution of future investment returns.

With these assumptions about preferences and the dynamic evolution of the investment opportunity set, the results of Merton (1971, 1973) and CIR (1985a,b) can be used to show that the representative investor selects a mean-variance efficient portfolio each instant. Since the assumption of logarithmic preferences implies that market weights are constant and that the risk-free rate is proportional to X , the continuous-time CAPM relation can be expressed as

$$\alpha_i = \lambda_0 + \lambda_1 \sigma_{iM}, \quad (3)$$

⁶ For example, see Fama (1981), Keim and Stambaugh (1986), Gibbons and Ferson (1985), and Fama and French (1988).

⁷ Assuming that Z_i (and Z_X in (2)) is a scalar Wiener process involves no loss of generality; if Z_i is a vector process, we can define a new scalar process by factoring the common \sqrt{X} term and summing the normally distributed components.

where λ_0 and λ_1 are constants and σ_{iM} is the constant portion of the instantaneous covariance of the return on firm i with the market's return.⁸ Thus, (3) places the familiar linear restriction on instantaneous expected returns; all cross-sectional variation in instantaneous expected returns is due entirely to cross-sectional variation in the instantaneous covariances with the market's return.

Because this restriction is imposed on the parameters of instantaneous returns, we need to express the restriction in terms of observable data in order to render it testable. In doing this, our approach is to identify the continuous-time parameters using the unconditional moments of returns measured over discrete intervals. First, we obtain an expression for temporally aggregated continuously compounded returns from the dynamics in (1) and (2) by applying Itô's Lemma, integrating the resulting dynamics, and applying Fubini's Theorem to change the order of integration. The mathematical details of this procedure are described in the Appendix. Next, we designate the unconditional mean, variance, covariance with the return on the market, and first-order autocovariance of the continuously compounded returns for firm i measured over an interval τ as M_i , V_i , C_i , and A_i , respectively. Using the temporally aggregated expression for returns in the Appendix and taking the appropriate expectations gives

$$M_i = \mu\tau(\alpha_i - \sigma_i^2/2), \tag{4}$$

$$V_i = \mu\tau\sigma_i^2 + \frac{2\mu\xi_1}{\kappa}(\alpha_i - \sigma_i^2/2)\sigma_{iX} + \frac{\mu\xi_2s^2}{\kappa^2}(\alpha_i - \sigma_i^2/2)^2, \tag{5}$$

$$C_i = \mu\tau\sigma_{iM} + \frac{\mu\xi_1\xi_3}{\kappa}\sigma_{iX} + \left(\frac{\mu s^2\xi_2\xi_3}{\kappa^2} + \frac{\mu\xi_1\sigma_{MX}}{\kappa}\right)(\alpha_i - \sigma_i^2/2), \tag{6}$$

$$A_i = \frac{\mu\xi_4^2}{\kappa^2}(\alpha_i - \sigma_i^2/2)\sigma_{iX} + \frac{\mu\xi_4^3s^2}{2\kappa^3}(\alpha_i - \sigma_i^2/2)^2, \tag{7}$$

where

$$\xi_1 = \tau - \frac{1 - e^{-\kappa\tau}}{\kappa}, \tag{8}$$

$$\xi_2 = \tau - \frac{2 - 2e^{-\kappa\tau}}{\kappa} + \frac{1 - e^{-2\kappa\tau}}{2\kappa}, \tag{9}$$

$$\xi_3 = \alpha_M - \sigma_M^2/2, \tag{10}$$

$$\xi_4 = 1 - e^{-\kappa\tau}, \tag{11}$$

σ_{iX} is the constant portion of the instantaneous covariance of the return on firm i with changes in the state variable, and α_M , σ_M^2 , and σ_{MX} are constants (market parameters).

These moments imply a number of interesting properties for discrete returns. For example, differentiating V_i with respect to τ shows that the variance of

⁸ The representative investor's first-order conditions imply $\alpha_i X = r + \lambda_1 \sigma_{iM} X$. However, $r = \lambda_0 X$, which implies (3).

returns does not grow linearly with the length of the holding period. In particular, the variance of returns can grow more slowly than implied by a random walk for small values of τ but can grow at a rate more rapid than implied by a random walk for larger values⁹ of τ . Another important property implied by these moments is that the value of the market beta for firm i is a function of the length of the period over which returns are measured. Thus, betas estimated from daily returns need not equal betas estimated from monthly data, all other estimation problems aside. Perhaps even more important, the relative ranking of firms by betas estimated from daily data need not be the same as the ranking based on betas estimated from monthly returns.¹⁰ This property underscores the importance of considering temporal aggregation in implementing tests of the CAPM. Finally, (7) shows that the first-order autocovariance of returns can be either positive or negative, depending on the values of the parameter σ_{iX} . If $\sigma_{iX} < 0$, then the return autocorrelation function can be close to zero for returns measured over short holding periods, decline to a minimum for intermediate holding periods, and then increase for long holding periods.

Equations (4), (5), (6), and (7) form a system of four equations in the four firm-specific instantaneous parameters α_i , σ_i^2 , σ_{iM} , and σ_{iX} . Inverting this system yields the following explicit expressions for the instantaneous parameters in terms of the unconditional moments of returns measured over discrete intervals of time:

$$\alpha_i = \frac{1}{\mu\tau} M_i + \frac{1}{2\mu\tau} V_i + \frac{\kappa\xi_1}{\mu\tau\xi_4^2} A_i + \frac{s^2}{2\mu^2\kappa^2\tau^3} (\xi_1\xi_4 - \xi_2) M_i^2, \quad (12)$$

$$\sigma_i^2 = \frac{1}{\mu\tau} V_i - \frac{2\kappa\xi_1}{\mu\tau\xi_4^2} A_i + \frac{s^2}{\mu^2\kappa^2\tau^3} (\xi_1\xi_4 - \xi_2^2) M_i^2, \quad (13)$$

$$\sigma_{iM} = \left(\frac{(\xi_1\xi_4 - 2\xi_2)\xi_3 s^2}{2\kappa^2\mu\tau^2} - \frac{\xi_1\sigma_{MX}}{\kappa\mu\tau^2} \right) M_i + \frac{1}{\mu\tau} C_i - \frac{\kappa\xi_1\xi_3}{\xi_4^2} \frac{A_i}{M_i}, \quad (14)$$

$$\sigma_{iX} = \frac{-\xi_4 s^2}{2\kappa\mu\tau} M_i + \frac{\kappa^2\tau}{\xi_4^2} \frac{A_i}{M_i}. \quad (15)$$

With these closed-form representations of the instantaneous parameters, we can now express the continuous-time CAPM in terms of the moments of discrete returns. From (3) and (4), the expected return on any firm is of the form:

$$M_i = \lambda_0 + \lambda_1 \sigma_{iM} + \lambda_2 \sigma_i^2, \quad (16)$$

where λ_2 is also a constant. Substituting the expressions for σ_i^2 and σ_{iM} from (13) and (14) into (16) and rearranging give a cubic equation for M_i . Solving for M_i

⁹ The intuition for this is that an asset's ability to hedge against shifts in the state variable (negative σ_{iX}) can dampen the growth rate of return variances for small τ , but the additional variance induced by the time-varying expected returns eventually offsets the dampening effect as τ increases.

¹⁰ This follows because C_i is a weighted combination of σ_{iM} , σ_{iX} , and $(\alpha_i - \sigma_i^2/2)$ and because the weights do not increase linearly as τ increases. For example, the rankings of C_i are governed primarily by the rankings of σ_{iM} for small τ . However, as τ increases, additional weight is given to the other idiosyncratic parameters, which need not have the same cross-sectional ranking as the σ_{iM} terms.

and taking the positive root result in a complicated closed-form expression for M_i as a nonlinear function of the three unconditional moments C_i , V_i , and A_i . This implies the surprising result that expected returns need not be linear in their market betas in the temporally aggregated CAPM.

Although the expression for M_i is nonlinear, the degree of nonlinearity is related to the length of the interval over which returns are measured. Over short intervals such as a month, M_i can be approximated¹¹ by the linear expression:

$$M_i = \gamma_0 + \gamma_1 C_i + \gamma_2 V_i + \gamma_3 A_i, \quad (17)$$

where γ_0 , γ_1 , γ_2 , and γ_3 are constants. Although the values of the γ coefficients depend on the parameters of the unobservable state variable dynamics, it is readily shown that $\gamma_0 > 0$, $\gamma_1 > 0$, and $\gamma_2 < 0$ for small τ . (The sign of γ_3 is indeterminate.) Assuming the restriction imposed on expected returns by the continuous-time CAPM to be linear greatly reduces the computational burden in performing the empirical tests. This linear version of the temporally aggregated model forms the basis of the empirical tests in Section III.

Note that (17) is a three-factor model of expected returns in the sense that all cross-sectional variation in expected returns is due to differences in just three variables. This is important because it implies that empirical tests which reject a single-factor model of expected returns cannot be interpreted as rejecting all forms of the CAPM. However, also note that two of the three factors in (17) are purely idiosyncratic; (17) is not equivalent to a model in which there are three common priced factors.

II. The Econometric Methodology

In the previous section, we derived a linear restriction imposed by the continuous-time CAPM on unconditional expected returns measured over discrete intervals. In this section, we propose a simple yet direct test of this linear restriction using observable returns.

Our econometric approach is to test (17) as a set of overidentifying restrictions on a system of moment equations using the Generalized Method of Moments (GMM) technique of Hansen (1982). This technique has a number of important advantages which make it an intuitive and logical choice for testing the continuous-time CAPM relation. First, the GMM approach does not require that the joint distribution of returns be multivariate normal; the asymptotic justification for the GMM procedure requires only that the joint distribution of returns be stationary and ergodic and that the relevant expectations exist. This feature is of particular importance in testing the continuous-time CAPM since the Appen-

¹¹ Using estimates of A_i from monthly returns shows that the cubic and constant terms in the expression for M_i are very small relative to the other terms for small τ . Dropping the cubic and constant terms and dividing the resulting quadratic expression by M_i result in (17). In order to determine how closely (17) approximates M_i for monthly returns, we used sample estimates of A_i , C_i , and V_i for the monthly returns on twenty size-based portfolios and the value-weighted market index during the 1966–1985 period and examined the approximation error implied by (17) for a wide range of values for the parameters κ and μ/σ^2 . The approximation error for the twenty portfolios was always less than one basis point per month and was generally much less.

dix shows that returns need not be multivariate normal in this time-varying framework. Second, the GMM estimators and their standard errors are consistent even if returns are conditionally heteroskedastic or if disturbances are serially correlated. French, Schwert, and Stambaugh (1987) have recently presented convincing evidence of the conditional heteroskedasticity of market returns. Finally, tests using the GMM technique can be implemented with long time series of returns since the asset pricing relation is expressed in terms of unconditional moments. This means that the parameters will be more precisely estimated than in conventional approaches,¹² which use only a limited time series of returns in order to avoid variation in the risk measures. The GMM technique has also been used in an asset pricing context by Hansen and Singleton (1982), Brown and Gibbons (1985), Ferson (1988), Harvey (1988), and others.

Define β to be the $3n + 4$ parameter vector with elements $\gamma_0, \gamma_1, \gamma_2, \gamma_3, C_1, C_2, \dots, C_n, V_1, V_2, \dots, V_n, A_1, A_2, \dots, A_n$, where n is the number of assets (or portfolios). Define the vector¹³ $h_t(\beta)$ as

$$h_t(\beta) = \begin{bmatrix} R_{1t} - \gamma_0 - \gamma_1 C_1 - \gamma_2 V_1 - \gamma_3 A_1 \\ \vdots \\ R_{nt} - \gamma_0 - \gamma_1 C_n - \gamma_2 V_n - \gamma_3 A_n \\ R_{1t}^2 - V_1 \\ \vdots \\ R_{nt}^2 - V_n \\ R_{1t} R_{Mt} - C_1 \\ \vdots \\ R_{nt} R_{Mt} - C_n \\ R_{1t} R_{1t-1} - A_1 \\ \vdots \\ R_{nt} R_{nt-1} - A_n \end{bmatrix} \tag{18}$$

Under the null hypothesis that the cross-sectional restriction in (17) is true, $E[h_t(\beta)] = 0$. The GMM procedure consists of replacing $E[h_t(\beta)]$ with its sample counterpart $g_T(\beta)$, where

$$g_T(\beta) = \frac{1}{T} \sum_{t=1}^T h_t(\beta), \tag{19}$$

¹² For example, see Fama and MacBeth (1973), Gibbons (1982), Stambaugh (1982), and Chan, Chen, and Hsieh (1985).

¹³ Following French, Schwert, and Stambaugh (1987), we do not adjust for the means in computing second moments because this adjustment is so small and significantly increases the computational difficulty of the estimation procedure. Correcting for the mean had little effect on the value of the weighting matrix.

and then choosing parameter estimates that minimize the quadratic form:

$$J_T(\beta) = g_T'(\beta)W_T(\beta)g_T(\beta), \quad (20)$$

where $W_T(\beta)$ is a positive-definite symmetric weighting matrix. Matrix differentiation shows that minimizing $J_T(\beta)$ with respect to β is equivalent to solving the homogeneous system of equations (orthogonality conditions):

$$D'(\beta)W_T(\beta)g_T(\beta) = 0, \quad (21)$$

where $D(\beta)$ is the Jacobian Matrix of $g_T(\beta)$ with respect to β and is independent of the data (nonstochastic) because of the structure of $g_T(\beta)$.

If $n = 4$, the parameters are just identified and $J_T(\beta)$ attains zero for all choices of the weighting matrix $W_T(\beta)$. In the more general case where $n > 4$, the GMM estimates of the overidentified parameter vector β depend on the choice of $W_T(\beta)$. Hansen (1982) shows that choosing $W_T(\beta) = S^{-1}(\beta)$, where

$$S(\beta) = E[h_T(\beta)h_T'(\beta)], \quad (22)$$

results in the GMM estimator of β with the smallest asymptotic covariance matrix.

As in Gibbons and Ramaswamy (1986) and Richardson and Smith (1988), the special structure of $g_T(\beta)$ implies that $S(\beta)$ does not depend on β ($S \equiv S(\beta)$). Thus, the GMM estimation can be conducted in a single step. Newey and West (1987) derive a consistent estimator of S which has the important property of being positive definite. Designating this estimator by S_0 , the asymptotic covariance matrix for the GMM estimate of β is

$$\frac{1}{T} (D_0'(\beta)S_0^{-1}D_0(\beta))^{-1}, \quad (23)$$

where D_0 is the Jacobian evaluated at the estimated parameters. This covariance matrix can be used to test the significance of the individual parameters. Note, however, that finding the coefficients γ_1 , γ_2 , and γ_3 to be significant (that is, finding that C_i , V_i , and A_i are priced) does not imply that the linear restriction is true. This follows since γ_1 , γ_2 , and γ_3 can be significant without explaining all cross-sectional variation in the unconditional means; the linear relation places a stronger restriction on expected returns.

The empirical tests of the continuous-time CAPM can be conducted as simple tests of the overidentifying restrictions. Under the null hypothesis that (17) is true, only $3n + 4$ parameters appear in the $4n$ moment equations (the vector $g_T(\beta)$). Hansen (1982) shows that T times the minimized value of $J_T(\beta)$ is asymptotically distributed as a $\chi^2(q)$ variate under the null hypothesis, where $q = n - 4$ is the number of overidentifying restrictions.

III. The Data

We implement the tests described in the previous section using CRSP monthly stock return data for the sixty-year period from 1926 to 1985. Because our

objective is to test a restriction on unconditional moments, we follow Chan and Chen (1988) and use long time series of returns to compute the appropriate sample moments. Accordingly, we organize the data into three twenty-year periods (1926–1945, 1946–1965, and 1966–1985) and test the linear restriction on mean returns within each of the individual periods. The overall test for the full sixty-year period can be conducted by aggregating the χ^2 test statistics from the individual twenty-year periods. (We assume that the test statistics are independent across the periods.)

In principle, the tests could be performed at an individual firm level since the unconditional sample moments of individual firms can be estimated to any desired degree of precision by using a sufficiently long time series of returns. However, this approach is computationally very difficult because of the size of the covariance matrix that is inverted in the GMM procedure. Consequently, we follow the traditional approach and conduct the tests at the portfolio level.

Several considerations guide our portfolio formation strategy. First, the portfolios should be formed in a manner that preserves cross-sectional differences in expected returns; otherwise, the tests will have little or no power. Accordingly, we form portfolios on the basis of firm size at the beginning of the period since firm size is well-known to be a useful instrumental variable for expected returns.¹⁴ Once formed, we would prefer to hold portfolio composition constant over the twenty-year test period. However, this long-term buy-and-hold strategy is not practical because a substantial proportion of the smaller firms are not listed for the full twenty-year period.¹⁵ As a result, attrition introduces bias¹⁶ into the sample moments unless the portfolios are periodically rebalanced. Consequently, we rebalance the size portfolios every five years as a tradeoff between portfolio continuity and attrition-induced bias. However, note that this approach implicitly assumes that the unconditional moments are unaffected by the rebalancing. Specifically, the portfolios are formed by sorting every firm listed in the CRSP monthly return file at the beginning of each five-year holding period into one of twenty portfolios. Portfolio returns are equally weighted averages of the returns on the firms contained within the portfolio.

Table I presents the average returns, value-weighted market betas, variances, and first-order autocorrelations for each of the portfolios over the study period. As shown, forming portfolios on the basis of firm size appears to give cross-sectional dispersion in all of the moments. Note that the decline in the autocorrelation coefficients from the portfolio with the smallest firms (portfolio 1) to the portfolio with the largest firms (portfolio 20) is gradual. Thus, cross-sectional differences in autocorrelations do not merely reflect the thin trading patterns of small firms discussed by Roll (1983). We use the value-weighted CRSP index in estimating the covariances C_i because it reflects the relative weighting of firms within the actual market portfolio.

¹⁴ For example, see Chan and Chen (1988).

¹⁵ Over fifty percent of the firms in the smallest size portfolio disappear from the sample in a typical twenty-year period.

¹⁶ For example, if the number of firms remaining in a portfolio is decreasing over time, then the portfolio is becoming less diversified and its variance is increasing. Thus, estimates of the unconditional variance of the portfolio's return will be meaningless.

Table I
Average Monthly Return, Market Beta, Variance, and
First-Order Autocorrelation by Size Portfolio, 1/1926 to
12/1985 (720 Observations)

All returns are continuously compounded. Average betas, variances, and autocorrelations for each size portfolio are averages of the estimates from the three twenty-year estimation periods. Market betas are computed using the CRSP value-weighted market index.

Size Portfolio ^a	Average Return	Market Beta	Variance	First-Order Autocorrelation
1	0.01532	1.281	0.01212	0.1766
2	0.01313	1.280	0.01017	0.1787
3	0.01283	1.261	0.00889	0.1492
4	0.01140	1.258	0.00843	0.1565
5	0.01200	1.260	0.00813	0.1837
6	0.01061	1.254	0.00754	0.1564
7	0.01074	1.206	0.00660	0.1663
8	0.01013	1.202	0.00660	0.1585
9	0.00973	1.167	0.00596	0.1475
10	0.00999	1.169	0.00585	0.1372
11	0.00943	1.184	0.00592	0.1546
12	0.00837	1.148	0.00591	0.1374
13	0.00916	1.139	0.00525	0.1444
14	0.00928	1.100	0.00475	0.1500
15	0.00892	1.128	0.00493	0.1426
16	0.00877	1.136	0.00548	0.1482
17	0.00896	1.049	0.00418	0.1304
18	0.00774	1.062	0.00446	0.1134
19	0.00692	1.003	0.00386	0.1050
20	0.00716	0.955	0.00314	0.0815

^a Portfolio 1 includes the smallest firms, portfolio 20 the largest.

IV. Empirical Results

In this section, we use the GMM framework to test the moment restrictions imposed by the linear asset pricing relation. We also test whether the unconditional variance and first-order autocovariance of returns provide incremental explanatory power for cross-sectional differences in expected returns after controlling for the unconditional covariance of returns with the market's return. We examine whether the differences between the sample means and the fitted values of the means (the residuals) are systematically related to portfolio size in order to determine whether the size effect is explained by the model. Finally, we also use the GMM framework to test the moment restrictions imposed by the discrete-time CAPM ($M_i = \gamma_0 + \gamma_1 C_i$).

A. Tests of the Continuous-Time CAPM

Using the continuously compounded returns on the twenty size portfolios, we obtain GMM estimates of the $3n + 4 = 64$ dimensional parameter vector, the covariance matrix of the GMM parameter estimates, and the χ^2 test statistic for the overidentifying restrictions for each of the three twenty-year periods. Under

the null hypothesis, only four parameters are required to explain the cross-sectional variation in unconditional mean returns; the continuous-time CAPM places $n - 4 = 16$ overidentifying restrictions on the system of moment equations. The minimization procedure is repeated with a variety of different starting values in order to ensure convergence to the global minimum.

Table II presents the χ^2 test statistics for the overidentifying restrictions. As shown, we cannot reject the linear moment restriction of the continuous-time CAPM in any of the three twenty-year periods. The p -values for the test statistics are, respectively, 0.762, 0.260, and 0.801. The test statistic for the full sixty-year study period is given by summing the individual χ^2 statistics and has a p -value¹⁷ of 0.714. Thus, the continuous-time CAPM appears to be well supported by the data even when confronted with returns from portfolios formed on the basis of size. We note that the critical five percent level of the test statistic for the overall period is 65.17. This indicates that the overall test statistic would have to be fifty-five percent larger before it could be rejected and provides some margin of error if the small sample distribution of the test statistic does not correspond precisely to its asymptotic distribution.¹⁸

The GMM parameter estimates of γ_0 , γ_1 , γ_2 , and γ_3 are also presented in Table II. As indicated, γ_3 is positive and highly significant in all three periods; the t -statistics for γ_3 are 2.72, 3.35, and 3.47, respectively. This shows that the information in the autocovariance A_i has significant cross-sectional explanatory power for unconditional expected returns. The positive sign of γ_3 (in conjunction with the positive values of A_i in Table I) implies that greater predictability on the basis of previous returns is associated with higher unconditional expected returns.

While the coefficients γ_0 , γ_1 , and γ_2 are not significant in any of the individual twenty-year periods, they are consistent in sign across the periods. Furthermore, the signs of γ_0 , γ_1 , and γ_2 are consistent with the signs predicted in Section I: $\gamma_0 > 0$, $\gamma_1 > 0$, and $\gamma_2 < 0$ in each of the three periods. We can examine the significance of the coefficients over the full sixty-year study period by examining the significance of the average t -statistic for the coefficients. The asymptotic t -statistics for these means are reported in the last row of Table II. As shown, the mean t -statistics for γ_0 and γ_1 are not significant at conventional levels. However, the mean t -statistic for γ_2 is significant at the 0.10 level. This provides weak evidence that the unconditional variance has cross-sectional explanatory power for mean returns.

B. The Continuous-Time CAPM and the Size Effect

As discussed in Section II, the tests of the overidentifying restrictions are tests against an alternative which places no cross-sectional restrictions whatsoever on

¹⁷ We also tested the overidentifying restrictions for the full sixty-year period directly by pooling the data. The resulting p -value for the restrictions was 0.846, which is consistent with the overall p -value reported in Table II. In addition, the estimated parameter values (0.0042, 3.34, -2.86, and 31.4) are also consistent with the averages of the subperiod parameter estimates. However, because of the difficulties associated with the pooling procedure (changing portfolio composition, strict parameter stationarity, etc.), the pooled results could potentially be less reliable and are not reported in the tables.

¹⁸ However, see Tauchen (1986).

Table II

Generalized Method of Moments Parameter Estimates and Tests of the Continuous-Time CAPM

$$M_i = \gamma_0 + \gamma_1 C_i + \gamma_2 V_i + \gamma_3 A_i$$

The parameters are estimated by the generalized method of moments using continuously compounded monthly return data for twenty size portfolios for each of the three twenty-year periods. The t -statistics for the parameters are in parentheses. M_i is the mean return for portfolio i , C_i is the covariance of the return for portfolio i with the CRSP value-weighted market index, V_i is the variance of the returns for portfolio i , and A_i is the first-order autocovariance of the returns for portfolio i . The test statistics (χ^2) for the individual periods are χ^2_{16} ; the test statistic is χ^2_{48} for the overall period. p -Values for the test statistics are in parentheses.

Period	γ_0	γ_1	γ_2	γ_3	χ^2
1926-1945	0.0014 (0.20)	2.49 (0.83)	-0.19 (-0.25)	6.10 (2.72)	11.74 (0.762)
1946-1965	0.0101 (1.03)	0.01 (0.30)	-5.59 (-1.53)	37.02 (3.35)	19.17 (0.260)
1966-1985	0.0084 (1.17)	1.89 (0.32)	-2.68 ^a (-1.28)	18.76 (3.47)	11.14 (0.801)
Overall ^a	— (1.39)	— (0.66)	— (-1.77)	— (5.51)	42.05 (0.714)

^a The t -statistics reported for the overall period are for the means of the t -statistics reported for the individual periods.

the unconditional expected returns of the portfolios (the just identified system). Accordingly, the hypothesis that expected returns are functions of firm size is embedded in the alternative hypothesis as a special case. Thus, the failure to reject the linear restriction imposed by the continuous-time CAPM is evidence against the existence of a size effect after properly controlling for risk.

However, in order to provide some information about the power of our tests of the overidentifying restrictions against the size alternative, we examine whether the difference between actual expected returns and the fitted expected returns from the estimation procedure (the residuals ϵ_i),

$$\epsilon_i = M_i - \gamma_0 - \gamma_1 C_i - \gamma_2 V_i - \gamma_3 A_i, \quad (24)$$

is systematically related to firm size. Accordingly, we regress the average values of ϵ_i over the sixty-year period on the natural logarithm of average firm size for each of the portfolios. The results of this regression are

$$\epsilon_i = -0.00027 - 0.000004 \ln Ave. Size_i, \quad (25)$$

(-0.79) (-0.15)

where $R^2 = 0.0007$ and the t -statistics (in parentheses) are based on the White (1980) heteroskedasticity-consistent estimate of the covariance matrix. This regression indicates that there is little or no relation between the logarithm of average firm size and the residuals from the unconditional continuous-time CAPM. This result is perhaps more directly demonstrated by simply plotting the residuals against the portfolio rankings. Figure 1 shows the unconditional expected return for each portfolio over the sixty-year period as well as the residuals. As illustrated, there is a pronounced relation between average size and unadjusted

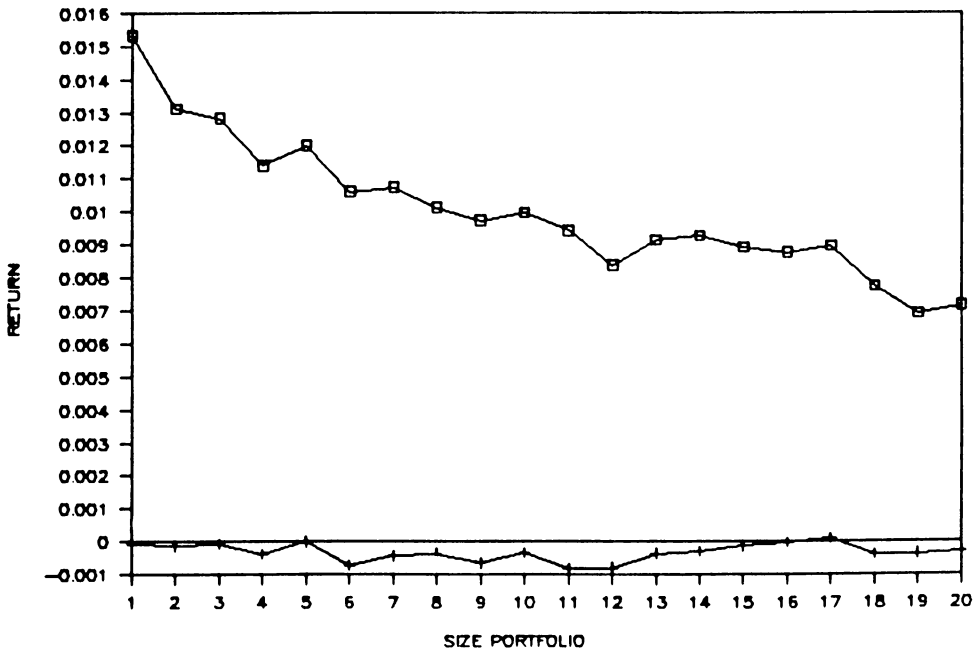


Figure 1. Average monthly return (labeled □) and average monthly residual or mispricing from the GMM estimation (labeled +) by size portfolio for the 1926–1985 period. Portfolio 1 contains the smallest firms, portfolio 20 the largest.

expected returns. However, this relation disappears after risk adjusting by the continuous-time CAPM.

C. Tests of the Discrete-Time CAPM

We can also conduct tests of the moment restrictions imposed by the discrete-time CAPM on unconditional means:

$$M_i = \gamma_0 + \gamma_1 C_i, \quad (26)$$

using the GMM methodology. In this expression, all cross-sectional variation in expected returns is due to cross-sectional differences in the covariance with the market's return (or, equivalently, the market beta). Thus, the discrete-time CAPM places $n - 2 = 18$ overidentifying restrictions on the system of moment equations.

Table III reports the GMM estimates of the parameters γ_0 and γ_1 and the associated χ^2 test statistics for the overidentifying restrictions for each of the periods. The results show that the discrete-time CAPM is easily rejected by the data; the discrete-time CAPM is strongly rejected in two of the three periods, and the p -value of the overall test statistic is only 0.0003. Again, note that the five percent level critical value of the overall test statistic is 72.15; the overall test statistic is thirty-four percent larger than the critical value.

These results also demonstrate the important point that the GMM procedure has power to reject a specific linear asset pricing relation at the unconditional

level. This provides evidence about the value of the GMM approach in testing asset pricing theories.

These tests are in the spirit of the unconditional tests in Chan and Chen (1988). However, our tests differ in several important ways. Chan and Chen test and fail to reject the discrete-time CAPM against the specific alternative of the log size-return relation. However, our tests examine a more general alternative. In addition, our tests use the value-weighted market index to compute the unconditional covariances with the market, while Chan and Chen use the equally weighted index. The combination of these two factors is the likely cause of the differences in results.

Table III shows that the unconditional covariance C_i has significant cross-sectional explanatory power for expected returns; the t -statistics for γ_1 for the three periods are 2.52, 3.68, and 2.80, respectively. Recall from the discussion in Section II that a rejection of the moment restrictions does not imply that C_i is without cross-sectional explanatory power. Interestingly, the estimates of the coefficient γ_0 are negative and have a significant mean t -statistic. This is consistent with the results of Chan and Chen, who also obtained negative estimates of γ_0 of the same order of magnitude.

Finally, we regress the residuals from the GMM estimation on the natural logarithm of average firm size as before. The resulting regression is

$$\epsilon_i = -0.00931 + 0.00031 \ln Ave. Size_i, \quad (27)$$

(-13.31) (4.98)

where R^2 is 0.571 and the t -statistics (in parentheses) are based on the White (1980) heteroskedasticity-consistent estimate of the covariance matrix. Thus,

Table III
Generalized Method of Moments Parameter Estimates and Tests of the Discrete-Time CAPM

$$M_i = \gamma_0 + \gamma_1 C_i$$

The parameters are estimated by the generalized method of moments using continuously compounded monthly return data for twenty size portfolios for each of the three twenty-year periods. The t -statistics for the parameters are in parentheses. M_i is the mean return for portfolio i , and C_i is the covariance of the return for portfolio i with the CRSP value-weighted market index. The test statistics (χ^2) for the individual periods are χ^2_{13} ; the test statistic is χ^2_{34} for the overall period. p -Values for the test statistics are in parentheses.

Period	γ_0	γ_1	χ^2
1926-1945	-0.0063 (-0.84)	16.50 (2.52)	37.10 (0.005)
1946-1965	-0.0150 (-2.01)	36.99 (3.68)	42.54 (0.001)
1966-1985	-0.0047 (-0.99)	8.06 (2.80)	17.13 (0.514)
Overall ^a	— (-2.22)	— (5.20)	96.76 (0.0003)

^a The t -statistics reported for the overall period are for the means of the t -statistics reported for the individual periods.

there is a strong relation between the residuals of the discrete-time CAPM and the proxy for average firm size.

V. Conclusion

We have examined the implications of the continuous-time CAPM for the properties of discretely observed returns. These implications differ significantly from those usually associated with the CAPM. For example, in the context of a simple model of conditional mean-variance efficiency, we have shown that the continuous-time CAPM becomes a multifactor model when the asset pricing relation is aggregated temporally. Tests of this multifactor relation at the unconditional level support the continuous-time CAPM but not the discrete-time, single-factor CAPM. In addition, the tests indicate that the information in the autocovariance function is useful in explaining the cross-sectional variation in expected asset returns.

These results have a number of important implications for asset pricing. First, the results suggest that models which allow for time-varying expected returns and risk measures (which introduce the autocovariance into the time-aggregated pricing relation) can lead to improved descriptions of return behavior. In addition, the GMM tests indicate that the size anomaly can be explained by equilibrium risk and return arguments even at the unconditional level. Finally, we have shown that empirical tests which reject a single-premium or -factor model of asset pricing cannot be interpreted as rejecting all forms of the CAPM.

Appendix

Let $Y_i = \ln P_i$. By Itô's Lemma and the dynamics in (1),

$$dY_i = (\alpha_i - \sigma_i^2/2)Xdt + \sigma_i\sqrt{X}dZ_i. \tag{A1}$$

Define the continuously compounded return for firm i from time 0 to τ as R_i . (Note that the dynamics in (A1) are time homogeneous.) The return R_i can be expressed in integral form as

$$R_i = (\alpha_i - \sigma_i^2/2) \int_0^\tau X(t) dt + \sigma_i \int_0^\tau \sqrt{X(t)} dZ_i(t). \tag{A2}$$

Now consider the dynamics of $e^{kt}X$. By applying Itô's Lemma and integrating,

$$X(t) = \mu + (X(0) - \mu)e^{-kt} + se^{-kt} \int_0^t e^{kt'} \sqrt{X(t')} dZ_X(t'). \tag{A3}$$

Substituting this expression into the first integral in (A2) gives

$$R_i = (\alpha_i - \sigma_i^2/2) \left(\mu\tau + \frac{(\mu - X(0))}{\kappa} (e^{-\kappa\tau} - 1) + s \int_0^\tau e^{-\kappa t} \int_0^t e^{kt'} \sqrt{X(t')} dZ_X(t') dt \right) + \sigma_i \int_0^\tau \sqrt{X(t)} dZ_i(t). \tag{A4}$$

However, a modified version of Fubini's Theorem (Ikeda and Watanabe (1981), Lemma 4.1 (p. 116)) allows us to express the double integral in (A4) as

$$\int_0^\tau e^{\kappa t'} \sqrt{X(t')} \int_{t'}^\tau e^{-\kappa t} dt dZ_X(t'), \quad (\text{A5})$$

which reduces to

$$\int_0^\tau \sqrt{X(t)} \frac{(1 - e^{-\kappa\tau} e^{\kappa t})}{\kappa} dZ_X(t). \quad (\text{A6})$$

Thus,

$$R_i = \frac{(\alpha_i - \sigma_i^2/2)}{\kappa} \left(\kappa\mu\tau + (\mu - X(0))(e^{-\kappa\tau} - 1) + s \int_0^\tau \sqrt{X(t)} (1 - e^{-\kappa\tau} e^{\kappa t}) dZ_X(t) \right) + \sigma_i \int_0^\tau \sqrt{X(t)} dZ_i(t). \quad (\text{A7})$$

Note that R_i involves two different stochastic integrals. Since $\sqrt{X(t)}$ is also stochastic, it is clear that neither of these two stochastic integrals will be normal variates. Hence, R_i will generally not be normally distributed. The unconditional moments can be obtained from (A7) by computing the conditional moments, which are linear in $X(0)$, and then taking the unconditional expectation.

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