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A NONLINEAR GENERAL EQUILIBRIUM MODEL OF THE TERM STRUCTURE OF INTEREST RATES

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We derive and test an alternative closed-form general equilibrium model of the term structure within the Cox, Ingersoll, and Ross theoretical framework in which yields are nonlinear functions of the risk-free rate. We show that equilibrium bond prices and the risk-free rate are not always inversely related and that bond risk need not be strictly increasing in maturity. Using Hansen's generalized method of moments to obtain parameter estimates, this nonlinear model outperforms the Cox, Ingersoll, and Ross square root model in describing actual Treasury bill yields for the 1964–1986 period.

1. Introduction

Modeling the term structure of interest rates has always been of fundamental importance to both financial economists and practitioners. In a recent paper, Cox, Ingersoll, and Ross (CIR) (1985a) develop a simple and intuitive general equilibrium framework for the pricing of discount bonds (the term structure) and other contingent claims in a continuous-time economy. An advantage of the CIR framework over partial equilibrium approaches[1] is that the risk-free rate and its dynamics are determined endogenously as part of the general equilibrium. Thus, interest-rate dynamics that permit arbitrage are precluded, a property that cannot always be guaranteed in a partial equilibrium setting.

In a companion paper, CIR (1985b) provide a specific example of their framework by deriving a closed-form model of the term structure in a simple one-state-variable linear production economy. Although innovative, this model [which we designate the square root (SR) model] does not capture fully the

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[1]For example, see Merton (1973), Vasicek (1977), Dothan (1978), Richard (1978), Brennan and Schwartz (1979), Langetieg (1980), Oldfield and Rogalski (1987), and Heath, Jarrow, and Morton (1987).

observed properties of the term structure. For example, it implies that term premiums are monotone increasing functions of maturity. Recent evidence in Fama (1984) and McCulloch (1987), however, suggests that actual term premiums have a humped pattern. In addition, the SR model allows only two types of yield curves (monotone or humped); observed yield curves frequently display more complicated patterns.

Clearly, we would like to be able to avoid these limitations while retaining the advantages and intuitive appeal of the CIR (1985a) general equilibrium approach to modeling the term structure. One possible way to do this is to introduce additional state variables into the analysis. Unfortunately, however, this approach is very costly in terms of tractability and the numbers of additional parameters that must be estimated.

This paper introduces a simpler and more direct way of addressing these limitations. By allowing technological change to affect production returns nonlinearly, we derive an alternative closed-form general equilibrium model of the term structure within the CIR (1985a) framework in which discount bond yields are nonlinear functions of the risk-free interest rate. This nonlinear dependence results in a richer set of yield curve and term premium shapes without introducing additional state variables or parameters. For example, this model [designated the double square root (DSR) model] is consistent with both humps and troughs in the yield curve as well as with monotone or humped patterns of term premiums.

Several new and surprising findings emerge from this nonlinear model. For example, we show that equilibrium discount bond prices and interest rates are not always inversely related; that discount bond prices can be increasing functions of the risk-free rate in some situations. This property has important implications for the invertibility of bond prices. In addition, we show that discount bond riskiness need not be a monotone increasing function of maturity or duration. Finally, we show that the local expectations hypothesis [CIR (1981)] can hold simultaneously for some bonds and not others.

To compare the empirical performance of the SR and DSR models in describing the term structure, we first use Hansen's (1982) generalized method of moments (GMM) technique to obtain point estimates of the parameters of the two models. We then compare the yields implied by each model with the actual yields on Treasury bills during the 1964–1986 period. We show that the DSR model has a lower root mean squared error than the SR model. In addition, the errors from the SR model are more serially correlated and more strongly related to the level of the risk-free rate. Finally, we test the yield nonlinearity property directly by examining whether changes in the square root of the interest rate explain yield changes after controlling for changes in the level of the risk-free rate. The results strongly support the hypothesis that yields are nonlinear functions of the risk-free rate, as implied by the model.

Section 2 begins by reviewing the basic CIR (1985a, b) framework and describing the alternative assumptions made in the DSR model. We assume that the state variable behaves as a random walk (with drift) over short periods, yet has a long-run steady-state distribution. This is consistent with the actual behavior of a number of economic variables [see Fama and French (1988)]. Following CIR (1985b), we derive the dynamics, conditional distribution, and unconditional distribution for the equilibrium interest rate. In section 3, we derive the equilibrium pricing function for discount bonds in the DSR model and present several examples of the types of yield curves and term premium patterns implied by the model. Section 4 discusses the GMM parameter estimation technique and presents the empirical results. Section 5 summarizes the major results of the paper and presents concluding remarks.

2. The double square root interest-rate model

In this section, we use the general equilibrium framework of CIR (1985a) to derive an explicit model of bond pricing and the term structure of interest rates. This model complements the SR model derived by CIR (1985b), which is also developed within the CIR (1985a) theoretical framework. Both models are rational expectation general equilibrium models in which the interest rate, interest-rate dynamics, bond prices, and bond-price dynamics are endogenous.

The CIR (1985a) general equilibrium framework can be summarized as follows. There is a finite number of constant stochastic returns to scale production technologies that produce a single good that can be allocated to either consumption or investment. There is also a fixed number of identical individuals who maximize a time-additive expected utility of consumption function by selecting optimal consumption and investment policies. For simplicity, all investment is done by firms and the individuals invest all of their unconsumed wealth in the shares of those firms. The values of the firms follow a multivariate diffusion process. Random technological change is introduced by allowing the drift vector and covariance matrix for this multivariate diffusion process to depend on a vector of state variables that is also governed by a multivariate diffusion process. The joint process for the firm values and the state variables completely describes the state of the system. There are perfect competitive markets for continuous trading in the firms' shares and a variety of contingent claims, as well as for instantaneous risk-free borrowing and lending. Equilibrium in this economy gives the market-clearing interest rate, prices for the contingent claims, and the total production and consumption plans.

To derive an explicit model of the term structure of interest rates from this framework, CIR (1985b) require some specific additional assumptions about production opportunities and preferences. For example, they assume that the representative investor has logarithmic preferences and that the single state variable governing changes in production and investment opportunities over time follows a square root process.

In deriving an alternative closed-form model for the term structure within the CIR (1985a) framework, we retain some of these additional assumptions, in particular, the assumption that the representative investor's preferences are logarithmic. As shown by Merton (1971), this assumption implies a convenient separability property for the derived utility of wealth function which simplifies the solution of the consumption and investment problem. In addition, we retain the assumption that technological change is governed by a single state variable, which we designate X.

Instead of assuming that X follows a square root process, however, we assume it follows a process that behaves locally as a random walk (with drift) but also has a long-run stationary distribution. These state-variable dynamics are intuitively reasonable and are consistent with the behavior of a variety of economic variables. For example, (log) stock prices resemble random walks with drift over short periods [see Fama (1976)], yet appear to have long-run stationary components as well [Fama and French (1988)]. Specifically, we assume that X is governed by the following stochastic differential equation (when $X > 0$):

$$\mathrm{d}X = m\,\mathrm{d}t + s\,\mathrm{d}Z, \tag{1}$$

where m and s are constants, $m < 0$, and Z is a standard Wiener process in R^1. When X reaches zero, the process returns immediately to positive values. This process is known as the reflected Brownian motion process[2] (with drift) and implies that the state variable is nonnegative and has the long-run stationary distribution

$$(-2m/s^2)\exp(2mX/s^2), \qquad X > 0, \tag{2}$$

which is the exponential density with mean $-s^2/2m$ and variance $s^4/4m^2$.

Finally, as in CIR (1985b), we assume that the means and variances of production returns are proportional. Rather than requiring them to be proportional to X (the linear case), however, we allow them to be proportional to the nonlinear term X^2. As shown by Sundaresan (1984), allowing technology to be nonlinear (for example, Cobb–Douglas) not only broadens the class of technologies available, but has the effect of inducing mean reversion in the equilibrium interest rate. This property is instrumental in developing a parsimonious model of the term structure.

[2] See Karlin and Taylor (1975, ch. 7) and Cox and Miller (1970, ch. 5) for a description of this process.

With these assumptions about production and the state-variable dynamics, we solve for the endogenous equilibrium interest rate. Because means and variances are proportional and preferences are logarithmic, optimal investment in each production process, optimal per capita consumption, and the representative investor's derived utility of wealth function are of the same form[3] as given in CIR (1985b). Thus the equilibrium interest rate is equal to the expected return on the market (optimally invested wealth) minus the variance of the market's return. Because market weights are constant, however, the expected return and variance of the market are proportional to X^2. Consequently,

$$r = cX^2, \tag{3}$$

where r is the instantaneous risk-free rate and c is a constant (assumed positive).

Applying Ito's Lemma to (3) gives the dynamics of the equilibrium interest rate,

$$\mathrm{d}r = (cs^2 + 2m\sqrt{c}\sqrt{r})\,\mathrm{d}t + 2s\sqrt{c}\sqrt{r}\,\mathrm{d}Z, \tag{4}$$

which can be rewritten as

$$\mathrm{d}r = \kappa(\mu - \sqrt{r})\,\mathrm{d}t + \sigma\sqrt{r}\,\mathrm{d}Z, \tag{5}$$

where $\kappa, \sigma > 0$ and $\mu = \sigma^2/4\kappa > 0$. This interest-rate process is similar to the SR process derived by CIR (1985b) and to the processes assumed by Vasicek (1977) and Brennan and Schwartz (1977), because the stochastic interest rate is elastically drawn toward a central value. The restoring force in (5) is proportional to $\mu - \sqrt{r}$, however, rather than to $\mu - r$, as in the other models of interest-rate dynamics. This nonlinear restoring force has many implications for the behavior of interest rates and bond prices that are discussed below. An analysis of the boundary behavior of the process indicates that the origin is a regular (attainable) boundary when $\kappa, \sigma^2 > 0$. On the other hand, ∞ is a natural (Feller) boundary that cannot be reached in finite time.[4] Because $\sqrt{r}$

[3] The representative investor's derived utility of wealth function is of the form $J(W, X, t) = A(t)\ln(W) + \xi(X, t)$, where W represents wealth. This in turn implies optimal consumption of the form $B(t)W$. Finally, as in CIR (1985b), market clearing implies that the vector of optimal weights can be expressed as

$$\Sigma^{-1}\alpha + \frac{1 - \mathbf{1}'\Sigma^{-1}\alpha}{\mathbf{1}'\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1},$$

which is constant, since both the vector of expected returns αX^2 and the covariance matrix of returns ΣX^2 are proportional to X^2 (α and Σ are constant). J_W/J_{WW} does not depend on $A(t)$ because of the functional form of J. Thus, as in CIR (1985b), the utility discount parameter ρ does not affect the equilibrium interest rate, which depends on J only through J_W/J_{WW}.

[4] The boundary behavior is determined by the values of four functionals related to the speed and scale measures of the process. See Karlin and Taylor (1981, ch. 15).

appears twice in (5), we designate this the double square root (DSR) interest-rate process.

The DSR process shares many of the empirically relevant properties of the CIR SR process:

(1) The singularity of the diffusion coefficient at $r = 0$ implies that negative interest rates are precluded.
(2) Because $\mu > 0$, the interest rate returns to positive values if it reaches zero.
(3) The instantaneous variance of the interest-rate process is $\sigma^2 r$; the instantaneous variance is directly related to the level of the interest rate.
(4) The interest rate has a stationary or steady-state distribution.

The DSR process also has the following interesting characteristics:

(5) Only two parameters are required to describe interest-rate dynamics, κ and σ^2.

This follows because μ^2 (the value toward which the interest rate reverts) is a function of the other two parameters, $\mu^2 = \sigma^4/16\kappa^2$. This means that the long-run fundamental interest rate cannot be specified independently of the parameters κ and σ^2; if σ^2 is higher in some regime, then (ceteris paribus) the mean interest rate is also higher, and vice versa.

(6) The rate at which the interest rate reverts towards μ^2 is asymmetric; interest rates are sticky downward.

This property arises because the drift of the process is proportional to $\mu - \sqrt{r}$. When $\sqrt{r} < \mu$ ($r < \mu^2$), the drift is upward; when $\sqrt{r} > \mu$ ($r > \mu^2$), the drift is downward. However, the drift is not as large (in absolute terms) when the interest rate is $\mu^2 + \varepsilon$ as when it is $\mu^2 - \varepsilon$. This asymmetry causes the interest rate to revert to μ^2 more slowly from above than from below.

The density of the interest rate at time t, r_t, conditional on its current value r, is obtained from the density of the square of a reflected Brownian motion,[5]

$$
\begin{aligned}
&\left(2\pi r_t \sigma^2 t\right)^{-1/2}\left[\exp\left(\frac{-2\left(\sqrt{r_t} - \sqrt{r} + \kappa t/2\right)^2}{\sigma^2 t}\right)\right. \\
&\left. + \exp\left(\frac{4\kappa\sqrt{r}}{\sigma^2}\right)\exp\left(\frac{-2\left(\sqrt{r_t} + \sqrt{r} + \kappa t/2\right)^2}{\sigma^2 t}\right)\right] \\
&+ \frac{2\kappa}{\sigma^2\sqrt{r_t}}\exp\left(\frac{-4\kappa\sqrt{r_t}}{\sigma^2}\right)\left(1 - \phi\left(\frac{2\left(\sqrt{r_t} + \sqrt{r} - \kappa t/2\right)}{\sqrt{\sigma^2 t}}\right)\right), \qquad (6)
\end{aligned}
$$

[5]The density of a reflected Brownian motion (with drift) is given in Cox and Miller (1970).

where $\phi(\cdot)$ is the cumulative standard normal distribution function. This expression implies that the conditional distribution of future interest rates is slightly skewed toward large values. The conditional moments of (6) are difficult to obtain in closed form. Simulations suggest, however, that both the conditional mean and the conditional variance are approximately linear in r. In the special case[6] where $\kappa = 0$ ($m = 0$), the interest rate is proportional to a noncentral χ^2 variate[7] and has conditional mean

$$\mathrm{E}[r_t | r] = r + \sigma^2 t/4 \tag{7}$$

and conditional variance

$$\mathrm{var}[r_t | r] = \sigma^2 (rt + \sigma^2 t^2/8), \tag{8}$$

where both conditional moments are linear in r.

In the general case ($\kappa > 0$), the distribution of the interest rate approaches a steady-state density as $t \to \infty$. The stationary density is

$$\frac{2\kappa}{\sigma^2 \sqrt{r}} \exp\left(\frac{-4\kappa\sqrt{r}}{\sigma^2} \right). \tag{9}$$

This is the Weibull[8] distribution with mean $\sigma^4/8\kappa^2$ and variance $5\sigma^8/64\kappa^4$. Because interest rates are sticky downward, the long-run mean value of the interest rate is $2\mu^2$; the drift or expected change in the rate can be negative even when the rate is below its long-run mean.

3. The double square root model of the term structure

We now derive the equilibrium price of a discount bond that pays one dollar at maturity. Because the state-variable and interest-rate dynamics are time-homogeneous, it is convenient to designate the current time as 0 and the payoff date of the bond as τ, which is the maturity of the bond as well.

Following CIR (1985b) [but using the state-variable dynamics in (1)], the fundamental valuation equation[9] for the discount bond $F(X, \tau)$ is

$$\frac{s^2}{2} F_{XX} + (m - \lambda X) F_X - rF - F_\tau = 0, \tag{10}$$

with the initial condition $F(X, 0) = 1$ and where λ is a constant[10] representing the market price of state-variable X risk. Because $X \geq 0$, however, the interest

[6] In this special case, the state variable follows a one-dimensional Bessel process. This is also known as radial Brownian motion or as a folded normal. See Karlin and Taylor (1975) and Leone, Nelson, and Nottingham (1961).

[7] See Johnson and Kotz (1970, ch. 28).

[8] See Johnson and Kotz (1970, ch. 20) for a description of the Weibull distribution and its properties.

[9] See CIR (1985a) for the definition of the fundamental valuation equation.

[10] As in CIR (1985b), the risk premium term λX represents the covariance of returns on optimally invested wealth with changes in the state variable. The instantaneous standard deviation

rate is a monotone, and therefore invertible, function of X. As do CIR (1985b), we use this property to make a change in variables from X to r, yielding the following transformed fundamental valuation equation for the price of a discount bond $P(r,\tau)$ [where $P(r,\tau) = F(X,\tau)$]:

$$\frac{\sigma^2}{2} r P_{rr} + \left(\frac{\sigma^2}{4} - \kappa\sqrt{r} - 2\lambda r \right) P_r - rP - P_\tau = 0, \tag{11}$$

with the initial condition $P(r,0) = 1$. By inspection of (11), the instantaneous expected return on a discount bond is $r + 2\lambda r P_r / P$; as in the CIR SR model, the instantaneous expected return is proportional to the bond's interest-rate elasticity. If λ is negative, then $P_r < 0$ implies positive term premiums.

A standard separation of variables approach gives the following equilibrium bond pricing function as the solution to (11):

$$P(r,\tau) = A(\tau)\exp\left(B(\tau)r + C(\tau)\sqrt{r}\right), \tag{12}$$

where

$$A(\tau) = \left(\frac{1-c_0}{1-c_0 e^{\gamma\tau}} \right)^{1/2} \exp\left(c_1 + c_2\tau + \frac{c_3 + c_4 e^{\gamma\tau/2}}{1 - c_0 e^{\gamma\tau}} \right),$$

$$B(\tau) = \frac{2\lambda - \gamma}{\sigma^2} + \frac{2\gamma}{\sigma^2(1 - c_0 e^{\gamma\tau})},$$

$$C(\tau) = \frac{2\kappa(2\lambda + \gamma)(1 - e^{\gamma\tau/2})^2}{\gamma\sigma^2(1 - c_0 e^{\gamma\tau})},$$

with

$$\gamma = \sqrt{4\lambda^2 + 2\sigma^2},$$

$$c_0 = (2\lambda + \gamma)/(2\lambda - \gamma),$$

$$c_1 = \frac{-\kappa^2}{\gamma^3\sigma^2}(4\lambda + \gamma)(2\lambda - \gamma),$$

$$c_2 = \frac{2\lambda + \gamma}{4} - \frac{\kappa^2}{\gamma^2},$$

$$c_3 = \frac{4\kappa^2}{\gamma^3\sigma^2}(2\lambda^2 - \sigma^2),$$

$$c_4 = \frac{-8\lambda\kappa^2}{\gamma^3\sigma^2}(2\lambda + \gamma).$$

of returns on wealth is proportional to X, whereas the instantaneous standard deviation of changes in the state variable is constant. Together this implies that the covariance between returns and changes in the state variable is proportional to X, where λ is the constant of proportionality.

Thus discount bond prices are functions of the two variables r and τ and depend parametrically on the constants κ, σ^2, and λ. The solution not only satisfies the initial condition $P(r,0)=1$, but implies economically reasonable bond prices as $r,\tau \to \infty$; the equilibrium bond pricing function satisfies the transversality conditions $P(r,\tau) \to 0$ as $\tau \to \infty$ and $P(r,\tau) \to 0$ as $r \to \infty$.

The most striking feature of the bond price is its nonlinear dependence on the interest rate; the yield to maturity of $P(r,\tau)$ is nonlinear in r. This property makes (12) unique as a closed-form solution for bond prices; the Merton (1973), Vasicek (1977), Richard (1978), Langetieg (1980), CIR (1985b), Oldfield and Rogalski (1987), and Heath, Jarrow, and Morton (1987) models all imply yields that are linear in r. The nonlinearity of yields in the risk-free rate is a directly testable empirical implication of the DSR model and will be examined in section 4.

The yield nonlinearity gives the equilibrium bond pricing function a number of interesting properties. For example, an increase in the interest rate need not always result in a decrease in the bond price; P_r is not uniformly negative. The intuition of this surprising result can be understood best by the following heuristic argument. Consider the properties of the risk-free rate as $r \to 0$. From (5), changes in the risk-free rate are deterministic in the limit. Furthermore, the drift of the process is positive if $\mu > 0$. Thus, an increase in r over the next instant becomes a certainty when $r=0$. If $P_r < 0$, then, ceteris paribus, a loss to the bondholder over the next instant is guaranteed, which is not consistent with equilibrium in the contingent claims market.

In actuality, P_r is not the sole determinant of the expected return on the bond; by Ito's Lemma, P_{rr} and P_τ are also related to the expected returns. This argument illustrates, however, that imposing the restriction $P_r < 0$ may not be consistent with equilibrium bond pricing. As $r \to 0$, $P_r > 0$ is necessary for the bond to have an equilibrium expected rate of return; at $r=0$, the expected return on the bond equals the risk-free rate.

By partial differentiation, we find that P_r is negative or positive as r is greater than or less than

$$\kappa^2(1-\mathrm{e}^{\gamma\tau/2})^4/\gamma^2(\mathrm{e}^{\gamma\tau}-1)^2, \tag{13}$$

which is generally a small value. Because the partial derivative of the equilibrium bond price with respect to r changes sign in the DSR model, however, the bond price is not a globally invertible function of the interest rate. This demonstrates that the commonly used technique of inverting bond prices[11] to solve for the state variables may not result in unique solutions for some

[11]For example, see Brennan and Schwartz (1979) and CIR (1985b).

equilibria. Partial differentiation also shows that bond prices are concave functions of the interest rate for small r, but become convex functions for larger values of r.

Straightforward computations show that $P_\tau < 0$ for all r and τ. This means that forward rates in the DSR model have the empirically realistic property of being uniformly positive. This property is shared by the CIR SR model, but some partial equilibrium models such as Merton (1973) and Ho and Lee (1986) [see Heath, Jarrow, and Morton (1987)] can imply negative forward rates.

The partial derivatives P_κ and P_{σ^2} can each take on positive or negative values. Intuitively, this is possible because changes in κ and σ^2 not only are changes in the speed of adjustment and the variance of the interest-rate process, but also affect the long-term average interest rate. Thus, changes in κ and σ^2 may have very complicated effects on bond pricing. Finally, the partial derivative P_λ is also ambiguous in sign. As we show later, this follows because term premiums for discount bonds are not necessarily monotone functions of maturity in the nonlinear DSR model.

From (12), the yield to maturity of a discount bond is

$$\frac{-1}{\tau}\left(\ln A(\tau) + B(\tau)r + C(\tau)\sqrt{r}\right), \tag{14}$$

which, as discussed earlier, is nonlinear in r. Because of this nonlinearity, the DSR model can lead to more complex and realistic yield curve shapes than term-structure models implying linear yields. To illustrate this, consider the CIR SR model, which uses three parameters to describe interest-rate movements and implies linear yields. CIR (1985b) show that the SR model can imply either a monotone or a humped yield curve, depending on parameter values. Fig. 1 shows, however, that the DSR model can be consistent with monotone term structures, term structures that increase, level out, then increase again, humped term structures, and term structures that have both a hump and a trough. Thus, the DSR model implies a richer and more realistic set of term-structure shapes.

Using l'Hopital's rule, we can show that the yield to maturity on a discount bond converges to r as $\tau \to 0$. As in the CIR SR model, the yield to maturity in the DSR model converges to a fixed value,

$$\kappa^2/\gamma^2 + (\gamma - 2\lambda)/4 > 0, \tag{15}$$

as $\tau \to \infty$, which is independent of the current interest rate. This is consistent with the well-known flattening of the yield curve for long-maturity bonds; if yields are determined by the risk-free rate r and there is a limiting yield to maturity as $\tau \to \infty$, then in the SR and DSR models the yields on long-term bonds should be less volatile over time than the yields on short-term bonds.

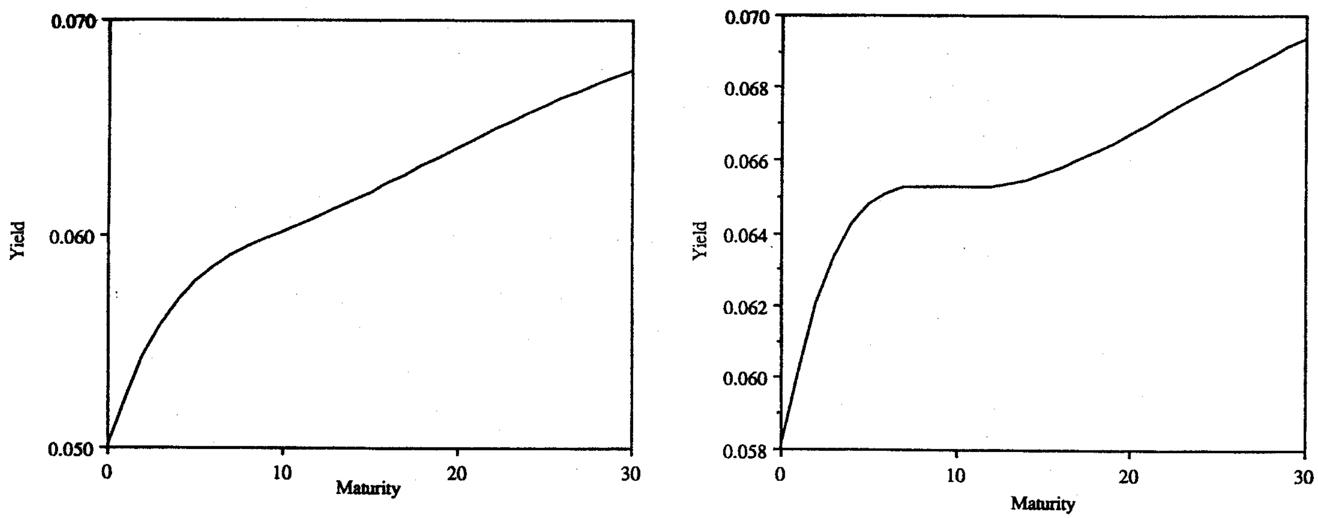

Fig. 1. Examples of yield curve shapes implied by the double square root term structure model using parameter values of $\kappa = 0.02$, $\sigma^2 = 0.03$, and $\lambda = -0.02$ for each graph and assuming an instantaneous risk-free rate of 0.05 and 0.058 per annum for the left and right panels, respectively, and 0.069 and 0.09 per annum for the left and right panels on the next page, respectively. The parameters κ and σ^2 govern the evolution of the instantaneous risk-free rate in the double square root model and the parameter λ represents the market price of interest rate risk. The yields are per annum and maturity is measured in years.

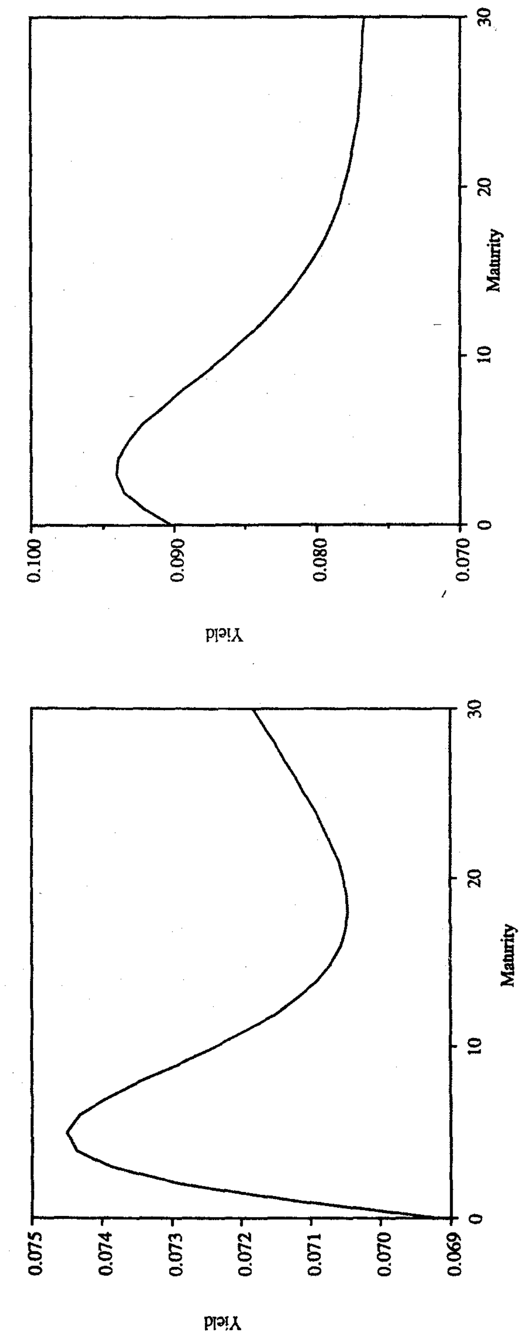

Fig. 1 (continued)

To illustrate this, we compute the standard deviation of month-to-month changes in the yields to maturity for one-month and twenty-year zero coupon Treasury securities for the May 1973–February 1987 period. The data are obtained from appendix II of Shiller and McCulloch (1987). The standard deviation of monthly yield changes for the one-month securities is 0.956%, whereas the same measure for the twenty-year securities is 0.383%. Thus, changes in the yields of the twenty-year securities are only about 40% as variable as those of the one-month securities. This order of magnitude is completely consistent with both the CIR SR model and the DSR model.

Applying Ito's Lemma to (12) gives the stochastic differential equation governing discount bond price dynamics,

$$\begin{aligned} \mathrm{d}P/P = {} & \left(r + 2\lambda\left(B(\tau)r + C(\tau)\sqrt{r}/2\right)\right)\mathrm{d}t \\ & + \left(B(\tau)\sqrt{r} + C(\tau)/2\right)\sigma\,\mathrm{d}Z. \end{aligned} \tag{16}$$

From (16), the instantaneous expected return on a discount bond converges to r as $\tau \to 0$. Furthermore, as $r \to 0$, the expected return also converges to zero.

Term premiums for discount bonds (expected returns minus r) in the DSR model are

$$2\lambda\left(B(\tau)r + C(\tau)\sqrt{r}/2\right). \tag{17}$$

Thus, term premiums depend on both τ and r; conditional term premiums need not equal unconditional premiums. Interestingly, term premiums in the DSR model need not be monotone increasing in τ. Fig. 2 shows that term premiums can either be monotone increasing or have a hump. Thus the DSR model is consistent with the results of Fama (1984), who finds that both ex ante and ex post premiums for Treasury bills have a hump at maturities of approximately nine months.[12] In contrast, the CIR SR model implies only term premiums that are monotone increasing in maturity. Fig. 2 also shows that term premiums in the DSR model, although positive for small τ, can actually become negative for larger τ. As $\tau \to \infty$, the term premium approaches the limit

$$\frac{2\lambda(2\lambda - \gamma)}{\sigma^2}\sqrt{r}\left(\sqrt{r} - \kappa/\gamma\right), \tag{18}$$

which is greater than or less than zero if r is greater than or less than κ^2/γ^2. Negative term premiums will occur if and only if $\kappa > \gamma\sqrt{r}$. If this condition is satisfied, it is straightforward to show that the instantaneous expected return

[12] However, see McCulloch (1987).

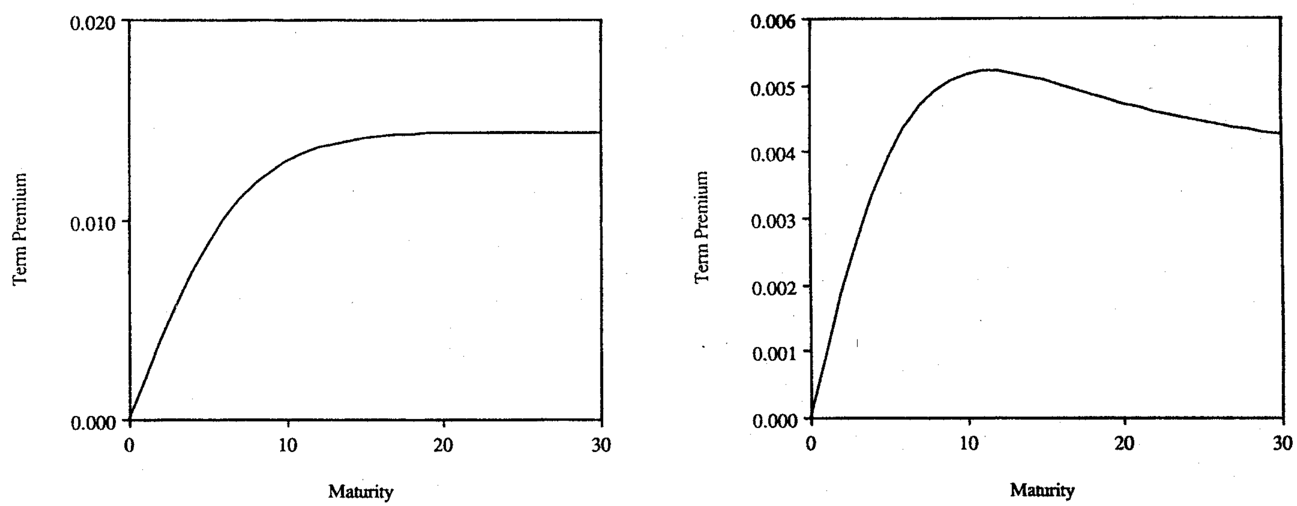

Fig. 2. Examples of term premium patterns implied by the double square root term structure model assuming parameter values $r = 0.10$, $\kappa = 0.005$, $\sigma^2 = 0.04$, and $\lambda = -0.01$ for the left panel, $r = 0.05$, $\kappa = 0.03$, $\sigma^2 = 0.03$, and $\lambda = -0.01$ for the right panel, and $r = 0.05$, $\kappa = 0.06$, $\sigma^2 = 0.03$, and $\lambda = -0.01$ for the panel on the next page. r denotes the per annum instantaneous risk-free rate, κ and σ^2 are parameters governing the evolution of the instantaneous risk-free rate in the double square root model, and the parameter λ represents the market price of interest-rate risk. The term premiums are per annum and maturity is measured in years.

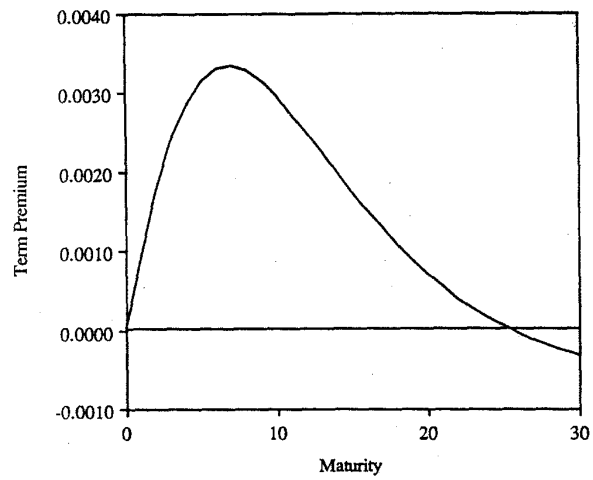

Fig. 2 (continued)

for the bond with maturity

$$\tau^* = \frac{2}{\gamma}\ln\left(\frac{\kappa+\gamma\sqrt{r}}{\kappa-\gamma\sqrt{r}}\right) \tag{19}$$

is equal to r. Consequently, when $\kappa > \gamma\sqrt{r}$, the local expectations hypothesis [see CIR (1981)] holds for the bond with maturity τ^*. This is consistent with market equilibrium, however, since from (16) the bond with maturity τ^* is locally risk-free; the diffusion term $(B(\tau)\sqrt{r} + C(\tau)/2)\sigma$ is zero for $\tau = 0$ and $\tau = \tau^*$. Thus locally risk-free bonds have expected returns equal to the risk-free rate in the DSR model.[13] Substituting τ^* for τ in (16) gives the following dynamics for the bond with maturity τ^*:

$$\mathrm{d}P/P = r\,\mathrm{d}t. \tag{20}$$

Thus, if an investor were to follow a portfolio strategy of holding the τ^* maturity bond, the return on the portfolio would be the same as if the investor rolled over instantaneously maturing bonds; the portfolio of the τ^* maturity bonds is immunized against basis risk in the sense of CIR (1980) and Ramaswamy and Sundaresan (1986).

Finally, the instantaneous variance of the bond returns is

$$\sigma^2\left(B^2(\tau)r + B(\tau)C(\tau)\sqrt{r} + C^2(\tau)/4\right). \tag{21}$$

[13]The converse is not true. When $r = 0$, the expected return on all bonds also equals zero, but the bonds may not be locally risk-free [the diffusion term in (16) need not equal zero when $r = 0$].

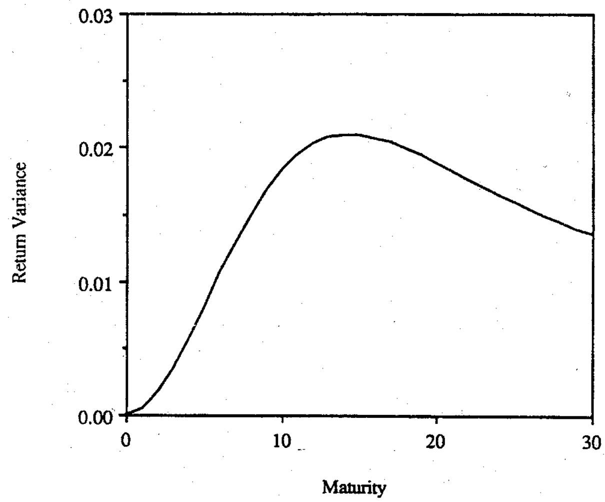

Fig. 3. Example of a nonmonotonic relation between discount bond return variance and maturity (duration) implied by the double square root term structure model assuming parameter values $r = 0.03$, $\kappa = 0.02$, $\sigma^2 = 0.015$, and $\lambda = -0.01$. r denotes the per annum instantaneous risk-free rate, κ and σ^2 are parameters governing the evolution of the instantaneous risk-free rate in the double square root model, and λ represents the market price of interest-rate risk. Returns are per annum and maturity is measured in years.

Because of the yield nonlinearity, the relation between the risk of bond returns and bond maturity[14] (duration) can be monotone increasing or can display more complicated patterns such as that shown in fig. 3.

4. An empirical comparison of the square root and double square root models

In this section, we compare the performance of the CIR (1985b) SR model with that of the DSR model in describing the actual yields on U.S. Treasury bills. After estimating the parameters of both models using Hansen's (1982) GMM technique, we compare the yields implied by the two models directly with observed Treasury bill yields. Finally, we test the yield nonlinearity property of the DSR model directly by examining whether changes in the square root of the risk-free rate explain significantly more about Treasury bill yield changes after controlling for changes in the level of the risk-free rate.

[14] Schaefer and Schwartz (1987) examine the relation between the standard deviation of returns on 87 U.K. government bonds and the duration of the bonds. Their evidence shows a strong monotone increasing relation between the two variables. They find the relation to be less strong, however, for low coupon bonds.

4.1. GMM parameter estimates

In general, estimating the parameters of continuous-time models from discretely sampled data is difficult and requires extensive computation. Because both the SR and the DSR models imply that the risk-free rate has a long-run stationary distribution, however, their parameters can be estimated directly by a simple application of the GMM technique. This approach consists of deriving analytical expressions for the unconditional expected yields of Treasury bills, setting these expressions equal to their sample counterparts, and then solving the resulting system of moment equations for the parameters' implied point estimates. In applying this technique, we use as many sample moments as parameters, which allows the parameter estimates to be uniquely determined. Note the similarity of this approach to that used in obtaining implied variance estimates from option prices; the main distinction is that we fit first moments instead of prices.

This empirical method has a number of advantages in estimating the parameters of the continuous-time interest-rate process. For example, the GMM approach uses the actual distribution of the temporally aggregated interest-rate process, unlike conventional approaches,[15] which often use a discrete-time approximation. In addition, the parameter estimates are unaffected by conditional heteroskedasticity and are robust to some forms of measurement error.[16] Finally, and perhaps most importantly, the technique is intuitive and easy to apply.

The equilibrium discount bond price in the CIR (1985b) SR model depends on the three interest-rate parameters κ, μ, and σ^2, as well as the market price of risk λ (these parameters need not have the same values as in the DSR model). To identify these four parameters, we first need to find four distinct moment equations, which can then be solved for the parameter values. From CIR (1985b, eqs. (19), (23), (25)), the expected yield for a τ maturity Treasury bill is

$$\frac{1}{\tau}(-\ln A(\tau) + B(\tau)\mu), \tag{22}$$

where

$$A(\tau) = \left(\frac{2\gamma e^{(\kappa+\lambda+\gamma)\tau/2}}{(\gamma+\kappa+\lambda)(e^{\gamma\tau}-1)+2\gamma}\right)^{2\kappa\mu/\sigma^2},$$

$$B(\tau) = \frac{2(e^{\gamma\tau}-1)}{(\gamma+\kappa+\lambda)(e^{\gamma\tau}-1)+2\gamma},$$

$$\gamma = \sqrt{(\kappa+\lambda)^2+2\sigma^2}.$$

[15]For example, see Brennan and Schwartz (1980).

[16]This feature is discussed by Gibbons and Ramaswamy (1986).

Taking τ to be the two-, three-, four-, and five-month[17] Treasury bills and setting the average yield for these maturities equal to the corresponding expressions from (22) gives a system of four equations that can be solved for the four unknowns κ, μ, σ^2, and λ.

Equilibrium discount bond prices in the DSR model depend on the two interest-rate parameters κ and σ^2 and on the market price of risk λ. Thus, three distinct moment equations are required to identify the model's parameters. Using the results in section 2 and the expression for yields in (14), it is readily shown that expected yields in the DSR model is given by

$$\frac{-1}{\tau}\left(\ln A(\tau) + B(\tau)\sigma^4/8\kappa^2 + C(\tau)\sigma^2/4\kappa\right). \tag{23}$$

Taking τ to be the three-, four-, and five-month Treasury bills[18] and setting the average yields for these maturities equal to the corresponding expression in (23) now gives a system of three equations that can be solved for the three parameters κ, σ^2, and λ.

Table 1 presents the GMM parameter estimates for the two models using monthly yield data for the June 1964–December 1986 period. The yields are obtained from the data set originally constructed by Fama (1984) and subsequently updated by the Center for Research in Security Prices (CRSP). These yields are based on the average of bid and ask prices for Treasury bills and are normalized to reflect a standard month of 30.4 days. All parameter estimates are annualized. Asymptotic standard errors for the parameter estimates are reported in parentheses and are computed from the positive definite Newey and West (1987) estimator of the covariance matrix of the moment equations [equivalent to the covariance matrix of the yields because of the linear structure of the moment equations as well as the separability of the data and the parameters – see Gibbons and Ramaswamy (1986)] and from the Jacobian matrix of the moments with respect to the parameters [see Hansen (1982) and Hansen and Singleton (1982)].

Although difficult to compare directly, the parameter estimates for the SR process in table 1 appear to be somewhat different from those estimated by Marsh and Rosenfeld (1983) or Dybvig and Brown (1986). Table 1 indicates that the GMM estimate of σ^2 is positive, in contrast to some of the estimates of the same parameter obtained by Brown and Dybvig (1986). The standard errors of the DSR model's parameters are somewhat large in relation to the point estimates, but this can be attributed mainly to the high correlation of the parameters with each other (> 0.98) rather than to any uncertainty about

[17] We use the one-month Treasury bill yield as a proxy for the short-term interest rate.

[18] The parameters were also estimated using the two-, three-, and four-month maturity yields. The resulting parameter estimates were very similar to those reported.

Table 1

Generalized method of moments estimates[a] of the parameters for the square root and double square root term-structure models using average yield to maturity data for two- to five-month Treasury bills from June 1964 to December 1986 (259 observations[b]).

		Square root model		
Parameter[c]	κ	μ	σ^2	λ
Point estimate	1.360	0.06660	0.00044	−0.487
Standard error[d]	0.062	0.102	0.00075	0.011
		Double square root model		
Parameter	κ	σ^2	λ	
Point estimate	0.00414	0.00306	−0.141	
Standard error[e]	0.024	0.017	0.383	

[a]The point estimates of the parameters are obtained as the implied values that set the sample means of the yields to maturity for two-, three-, four-, and five-month Treasury bills (three-, four-, and five-month Treasury bills for the double square root model) equal to their corresponding unconditional first moments.

[b]Yield data are missing for a few months for some of the Treasury bills. As in Fama (1984), these months are deleted for all maturities.

[c]The parameters κ, μ, σ^2, and λ are the four parameters used in the Cox, Ingersoll, and Ross (1985b) square root term structure model. Similarly, the parameters κ, σ^2, and λ in the second half of the table are the parameters used in the double square root term structure model. The parameter values for the two models need not be the same.

[d]The standard errors are computed using the Newey and West (1987) heteroskedasticity- and autocorrelation-consistent estimate of the covariance matrix of the yields [which is identical to the covariance matrix of the moment equations because of the structure of the moment equations (yields − mean yield)].

[e]The larger standard errors for the double square root model are primarily due to the high correlations among the individual parameters.

the values of the point estimates. (The point estimates are the unique solutions of the system of moment equations.) The probable effect of the larger standard errors for the DSR model will be to bias the tests against the model.

4.2. Yield comparisons

With these parameter estimates it is now possible to compute theoretical yields for longer-maturity U.S. Treasury bills from both the SR and the DSR models. First, we compare the yields from the SR model directly with those of the DSR model to determine how different the models are in their pricing implications. We then compare these theoretical yields with the actual yields on U.S. Treasury bills[19] with maturities of six to twelve months during the

[19]The risk-free rate and the parameter estimates are obtained from Treasury bills with maturities of one to five months. Thus, comparing the yields implied by the two models for six- to twelve-month Treasury bills provides an out-of-sample test. The lack of a sufficiently long time series of data for longer-maturity discount bonds rules out tests using maturities beyond twelve months. Coupon bond data are not appropriate for the tests because tax-trading-related options in coupon bond prices could induce bias [see Constantinides and Ingersoll (1984)].

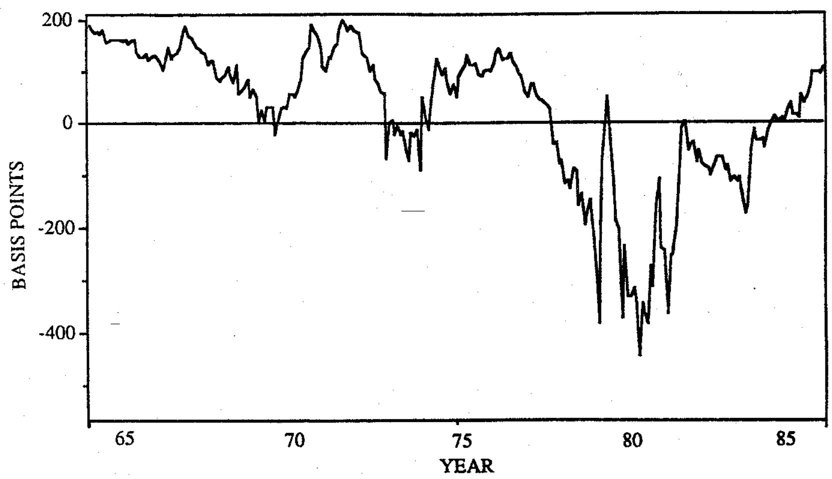

Fig. 4. Difference between the yields implied by the Cox, Ingersoll, and Ross (1985b) square root model and those implied by the double square root model for twelve-month Treasury bills for the June 1964–December 1986 period (monthly). The differences are expressed as basis points per annum.

June 1964–December 1986 period to see which model best captures the observed term structure. As before, yields to maturity for U.S. Treasury bills are obtained from the data set originally used by Fama (1984).

Fig. 4 displays the difference between the yield to maturity of a twelve-month Treasury bill implied by the SR model and the corresponding yield implied by the DSR model. The differences between the two models can be quite large. The average absolute difference during the study period is more than 112 basis points, and differences of more than 250 basis points occurred several times during 1980 and 1981. These simulation results demonstrate that the two models have fundamentally different implications for the behavior of equilibrium discount bond prices. The yields implied by the DSR model are generally less than those implied by the SR model from 1964 to 1978, but the relation is reversed for the 1979–1985 period. Interestingly, the average difference in the yields implied by the two models over the entire 1964–1986 study period is only about 22 basis points.

Table 2 presents summary statistics for the differences between yields implied by the two models and actual yields for six- to twelve-month Treasury bills from June 1964 to December 1986. The DSR model appears to capture the variation in the term structure better than the SR model; the root mean squared error (RMSE) of the DSR model is uniformly less than the RMSE of the SR model for all maturities. Decomposing the RMSE into its components

Table 2

Summary statistics for the differences between theoretical yields and actual U.S. Treasury bill yields (in basis points) using monthly data from June 1964 to December 1986 (259 observations[a]).

	Maturity in months						
	6	7	8	9	10	11	12
	Square root model yield[b] *– Actual yield*						
RMSE[c]	80.4	86.5	92.9	99.7	107.3	112.9	117.5
Mean	7.2	12.4	16.4	19.1	26.4	32.5	37.2
Std. dev.	80.1	85.6	91.4	97.9	104.0	108.1	111.5
Corr. with r	−0.733	−0.736	−0.756	−0.770	−0.787	−0.800	−0.805
ρ[d]	0.755	0.755	0.787	0.816	0.842	0.861	0.869
	Double square root model yield[b]*–Actual yield*						
RMSE	58.6	62.9	66.0	69.8	72.7	76.5	81.9
Mean	−15.8	−11.6	−9.1	−6.0	1.4	8.3	14.6
Std. dev.	56.4	61.8	65.4	69.5	72.7	76.4	80.6
Corr. with r	0.259	0.348	0.400	0.438	0.470	0.521	0.571
ρ[d]	0.260	0.266	0.313	0.377	0.417	0.465	0.510

[a]Yield data are missing for a few months for some of the Treasury bills. As in Fama (1984), these months are deleted for all maturities.
[b]The square root model and double square root model yields are computed using the one-month Treasury bill rate as a proxy for the short-term interest rate.
[c]RMSE is the root mean squared error.
[d]ρ represents the first-order autocorrelation.

shows that both the bias and the standard deviation of differences between yields implied by the DSR model and actual yields (the errors) are markedly less than the corresponding measures for the SR model.

Table 2 also gives the correlation between the two models' errors and the level of the risk-free rate. The SR model's errors are strongly negatively related to r; the correlations range from -0.733 for the six-month Treasury bill to -0.805 for the twelve-month bill. The DSR model's errors are less correlated with the risk-free rate, with correlations ranging from 0.259 for the six-month Treasury bill to 0.571 for the twelve-month bill. These correlations provide some intuition for the large differences between the two models shown in fig. 4. Since the SR model's errors are negatively correlated with the risk-free rate and the DSR model's errors are positively correlated, the differences between the two tend to be larger than the errors themselves. The correlation between the two model's errors and the risk-free rate may also provide a partial explanation for the persistence of the errors over time. The first-order autocorrelation coefficients for the SR model's errors range from 0.755 for the six-month Treasury bill to 0.869 for the twelve-month bill. Although the DSR model's errors are also autocorrelated, the first-order autocorrelation coefficients are only about one half as large as those for the SR model's errors.

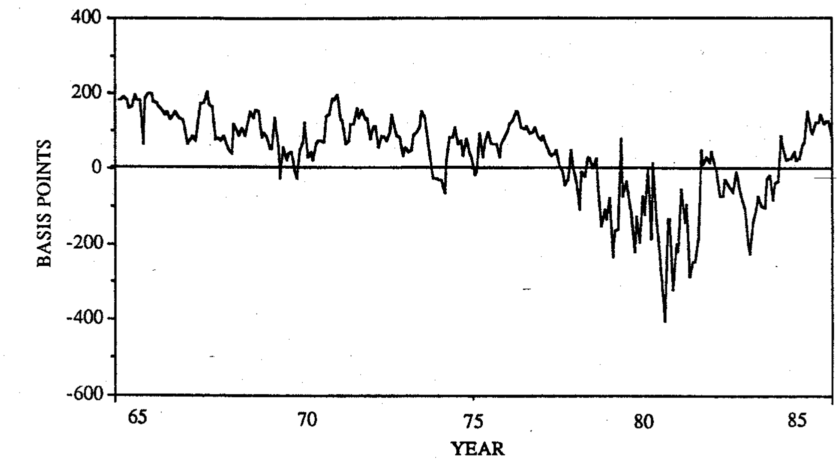

Fig. 5. Difference between the yields implied by the Cox, Ingersoll, and Ross (1985b) square root model and the actual yields on twelve-month U.S. Treasury bills for the June 1964–December 1986 period (monthly). The differences are expressed as basis points per annum.

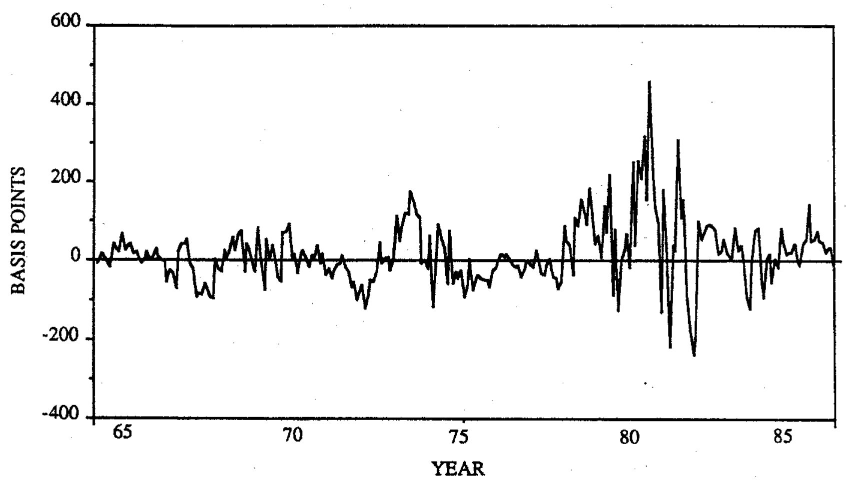

Fig. 6. Difference between the yields implied by the double square root model and the actual yields on twelve-month U.S. Treasury bills for the June 1964–December 1986 period (monthly). The differences are expressed as basis points per annum.

To illustrate the behavior of the errors over time, fig. 5 plots the SR model's errors for the twelve-month Treasury bill for the 1964–1986 period and fig. 6 the DSR model's errors. The SR model's errors are quite variable, and the degree of bias tends to vary appreciably over time. The DSR model's errors are less variable, and the bias appears to be fairly small and constant throughout the period.

Neither model completely captures the level and variation of Treasury bill yields during the study period. This is best seen by noting that the root mean squared error of the models is related to maturity in table 2; the RMSE of the DSR model increases by 39.8% as maturity doubles from six to twelve months and the RMSE of the SR model increases by 46.1%. Extrapolating this trend to longer maturities suggests that models that incorporate an additional maturity-related factor [as in Brennan and Schwartz (1979)] could lead to an improved description of long-maturity discount bond yields. Alternatively, these trends indicate that the models perform best when the parameters are estimated from data similar in maturity to the yields being modeled.

4.3. *Empirical tests of yield nonlinearity*

We now compare the two models' success in capturing changes in Treasury bill yields over time. Recall that the SR model (along with many others) implies that yields are linear functions of the risk-free rate. Thus, changes in yields are linearly related to changes in the risk-free rate. In contrast, the DSR model implies that yields depend on both r and $\sqrt{r}$, so yield changes are linearly related to changes in both r and $\sqrt{r}$. Consequently, the DSR model can be tested simply by examining whether changes in $\sqrt{r}$ have incremental explanatory power for yield changes after controlling for the change in r.

The yield nonlinearity implies that the DSR model can be viewed as a two-factor model of the term structure. This follows because yields are linear functions of r and $\sqrt{r}$ in the DSR model and r and $\sqrt{r}$ are linearly[20] independent. Together, these properties imply that two (linear) factors are required to span the space of yields. Equivalently, the yield on a discount bond can be expressed as a linear function of any other two bonds. Thus, the DSR model has the potential to capture the multifactor[21] nature of the term structure while retaining the analytical tractability of a one-state-variable model.

We designate the yield to maturity on a τ-month Treasury bill at the end of month t as $Y_{\tau t}$. Subtracting $Y_{\tau t-1}$ from $Y_{\tau t}$ gives the change in yield for a

[20] Two functions are linearly independent if their Wronskian is nonzero. The Wronskian of r and $\sqrt{r}$ is $-\sqrt{r}/2 \neq 0$. See Birkhoff and Rota (1978) for a description of the Wronskian.

[21] For example, see Brennan and Schwartz (1980, 1982), Oldfield and Rogalski (1981), and Stambaugh (1988).

constant-maturity Treasury bill, $\Delta Y_{\tau t}$. This involves two different τ-month Treasury bills; we hold maturity (not Treasury bills) constant in computing the yield changes. Because of this, (14) implies that $\Delta Y_{\tau t}$ can be expressed as

$$\Delta Y_{\tau t} = \beta_0 + \beta_1 \Delta r_t + \beta_2 \Delta\sqrt{r_t} + \varepsilon_t, \tag{24}$$

where $\beta_0 = 0$, β_1 and β_2 are constants (because τ is held constant), $\Delta r_t = r_t - r_{t-1}$, $\Delta\sqrt{r_t} = \sqrt{r_t} - \sqrt{r_{t-1}}$, and ε_t is an error term. We include an error term because of the possibility of measurement errors in Treasury bill prices arising from quotation errors or from the averaging[22] of bid and ask prices. We assume that the error term is normally distributed with mean zero. Hence (24) is a well-specified regression equation. We allow the residual ε_t to be heteroskedastic. In addition, we permit the residuals to be either independent over time or to follow an AR(1) process, and to be contemporaneously correlated across the regression equations for Treasury bills of different maturities. Test of whether yields are nonlinear in the risk-free rate can be conducted simply as tests of the null hypothesis $\beta_2 = 0$.

One advantage of this testing approach is that it does not require estimates of the parameters κ, σ^2, and λ. Thus, inferences about yield nonlinearity are unaffected by any estimation error that might be present in the parameters given in table 1. As a result, these tests provide a useful complement to the comparisons of the previous section, which do depend on the parameter estimates.

Table 3 presents summary statistics for the monthly changes in yields for the six- to twelve-month Treasury bills for June 1964–December 1986, $\Delta Y_{6t}, \Delta Y_{7t}, \ldots, \Delta Y_{12t}$. Summary statistics for Δr_t and $\Delta\sqrt{r_t}$ are also presented. The monthly changes in yields are approximately mean zero over the study period. The yield changes for the six- to twelve-month Treasury bills are less volatile than those for the one-month Treasury bill, again illustrating the dampening of yield volatility with maturity. In addition, ΔY_{6t} through ΔY_{12t} have a trace of positive first-order autocorrelation, whereas the opposite is true for Δr_t and $\Delta\sqrt{r_t}$. This difference in the time-series properties of the yield changes and Δr_t is itself an indication that yields are not simple linear functions of the risk-free rate. The last two columns of table 3 present the pairwise (univariate) correlations between the yield changes and Δr_t and $\Delta\sqrt{r_t}$. The correlation of each yield spread with $\Delta\sqrt{r_t}$ is higher than the correlation

[22] Using first differences of yields eliminates most of the error arising from the averaging of bid and ask prices; $\Delta Y_{\tau t}$ computed by averaging bid and ask prices is essentially the same as $\Delta Y_{\tau t}$ computed using only bid prices or only ask prices. This is because the bid–ask spread is much less variable than the bond prices and essentially acts as a constant in the differencing. See McCulloch (1987).

Table 3

Summary statistics for monthly changes in Treasury bill yields, the risk-free rate, and the square root of the risk-free rate from June 1964 to December 1986 (258 observations[a]).

Variable[b]	Mean[c]	Std. dev.[d]	S.R.[e]	ρ_1[f]	ρ_2	ρ_3	Corr.[g] Δr	Corr.[h] $\Delta\sqrt{r}$
ΔY_6	8.344	7.074	9.64	0.144	−0.065	−0.094	0.692	0.700
ΔY_7	8.386	6.938	9.56	0.128	−0.036	−0.094	0.630	0.645
ΔY_8	8.334	6.880	9.63	0.157	−0.042	−0.130	0.619	0.632
ΔY_9	8.199	6.988	10.09	0.161	−0.074	−0.135	0.620	0.631
ΔY_{10}	8.190	6.876	10.17	0.158	−0.069	−0.125	0.617	0.628
ΔY_{11}	8.283	6.836	10.24	0.163	−0.077	−0.119	0.624	0.633
ΔY_{12}	8.041	6.885	10.35	0.160	−0.075	−0.119	0.622	0.632
Δr	6.586	8.849	11.19	−0.126	0.057	−0.108	1.000	0.984
$\Delta\sqrt{r}$	16.460	14.42	9.88	−0.085	0.051	−0.104	0.984	1.000

[a] Yield data are missing for a few months for some of the Treasury bills. As in Fama (1984), the months are deleted for all maturities. Changes in the yields are computed over successive observations.

[b] ΔY_τ represents the monthly change in the τ-month Treasury bill yield. Δr represents the monthly change in the risk-free rate and $\Delta\sqrt{r}$ represents the monthly change in the square root of the risk-free rate. All yields are continuously compounded, annualized, and expressed in decimal form.

[c] Multiplied by 100,000.

[d] Multiplied by 1,000.

[e] Standardized range.

[f] ρ_i is the sample autocorrelation for lag i.

[g] Represents the correlation of the indicated variable with Δr.

[h] Represents the correlation of the indicated variable with $\Delta\sqrt{r}$.

with Δr_t, although the difference is small in absolute terms, ranging from 0.008 to 0.015.

The empirical tests are conducted by regressing the yield changes on the contemporaneous changes in the risk-free rate and in the square root of the risk-free rate in individual (univariate) regressions. We estimate the regressions individually rather than in a multivariate framework such as Zellner's (1962) seemingly unrelated regression (SUR) model because each regression has the same explanatory variables; ordinary least squares (OLS) and SUR yield identical results in this situation.[23]

Table 4 presents the results of the regressions estimated by OLS (residuals are assumed to be serially uncorrelated). The t-statistics (in parentheses) are based on the White (1980) heteroskedasticity-consistent estimate of the covariance matrix. The results in table 4 uniformly appear to support the yield nonlinearity implications of the DSR model. The estimate of β_2 is significant at the 0.05 level for five of the seven regressions estimated and at the 0.075

[23] See Judge, Griffiths, Hill, Lutkepohl, and Lee (1985, ch. 2).

Table 4

Ordinary least squares regression tests of yield nonlinearity using monthly changes in Treasury bill yields from June 1964 to December 1986 (258 observations[a]).

$$\Delta Y_{jt} = \beta_0 + \beta_1 \Delta r_t + \beta_2 \Delta\sqrt{r_t} + \varepsilon_t, \quad j = 6, 7, \ldots, 12.^{\text{b}}$$

Maturity (months)	β_0[c]	β_1	β_2	$D.W.$	Adj. R^2
6	0.0294 (0.09)[d]	0.0743 (0.28)	0.2985 (2.13)	2.44	0.486
7	0.0285 (0.09)	−0.1267 (−0.47)	0.3869 (2.66)	2.49	0.413
8	0.0316 (0.10)	−0.0651 (−0.23)	0.3405 (2.24)	2.40	0.394
9	0.0304 (0.09)	−0.0397 (−0.13)	0.3297 (2.05)	2.33	0.394
10	0.0316 (0.09)	−0.0322 (−0.11)	0.3187 (2.02)	2.32	0.390
11	0.0349 (0.10)	0.0415 (0.14)	0.2747 (1.79)	2.29	0.396
12	0.0314 (0.09)	0.0173 (0.06)	0.2910 (1.82)	2.29	0.394

[a]Yield data for some of the maturities are not available for a few of the months. As in Fama (1984), these months are deleted for all maturities. Changes in the yields are based on successive observations.

[b]ΔY_{jt} is the change during month t of the yield to maturity on j-month Treasury bills (holding maturity and not bill constant). Δr_t is the change in the one-month Treasury bill rate during month t and $\Delta\sqrt{r_t}$ is the change in the square root of the one-month Treasury bill rate during month t.

[c]Multiplied by 1,000.

[d]t-statistics in parentheses are based on the White (1980) heteroskedasticity-consistent estimate of the covariance matrix.

level for all seven. The t-statistics for the β_2 estimates decline from 2.66 for the seven-month Treasury bills to 1.79 for the eleven-month bills.

Surprisingly, the estimates of β_1 are never significantly different from zero, and are all within one-half of a standard deviation from zero. This is not necessarily evidence, however, that $\Delta\sqrt{r_t}$ subsumes Δr_t in explaining yield changes; the high correlation between the two variables may cause the t-statistic for β_1 to appear smaller than would otherwise be the case. Finally, the estimates of β_0 are all insignificantly different from zero, as expected. The adjusted R^2 for the regressions ranges from about 0.49 for the six-month Treasury bills to approximately 0.39 for the eight- to twelve-month bills.

Some of the Durbin–Watson statistics for the regression (in particular, those for the six-, seven-, and eight-month maturities) suggest a trace of negative autocorrelation in the residuals. Since the standard errors of the coefficient estimates could be biased in the presence of autocorrelation, we

Table 5

Cochrane–Orcutt regression tests of yield nonlinearity using monthly changes in Treasury bill yields from June 1964 to December 1986 (258 observations[a]).

$$\Delta Y_{jt} = \beta_0 + \beta_1 \Delta r_t + \beta_2 \Delta\sqrt{r_t} + \varepsilon_t, \quad j = 6, 7, \ldots, 12.^{\text{b}}$$

Maturity (months)	β_0[c]	β_1	β_2	$D.W.$	Adj. R^2
6	0.0303 (0.10)[d]	0.2261 (0.99)	0.2531 (2.09)	2.16	0.569
7	0.0265 (0.09)	0.0319 (0.13)	0.3472 (2.75)	2.19	0.522
8	0.0294 (0.10)	0.0575 (0.23)	0.3139 (2.28)	2.16	0.483
9	0.0309 (0.10)	0.0566 (0.20)	0.3094 (2.08)	2.13	0.461
10	0.0329 (0.10)	0.0524 (0.19)	0.3033 (2.06)	2.11	0.453
11	0.0349 (0.11)	0.1104 (0.41)	0.2646 (1.83)	2.11	0.451
12	0.0311 (0.09)	0.0906 (0.32)	0.2783 (1.84)	2.11	0.449

[a]Yield data for some of the maturities are not available for a few of the months. As in Fama (1984), these months are deleted for all maturities. Changes in the yields are based on successive observations.

[b]ΔY_{jt} is the change during month t of the yield to maturity on j-month Treasury bills (holding maturity and not bill constant). Δr_t is the change in the one-month Treasury bill rate during month t and $\Delta\sqrt{r_t}$ is the change in the square root of the one-month Treasury bill rate during month t.

[c]Multiplied by 1,000.

[d]t-statistics in parentheses are based on the White (1980) heteroskedasticity-consistent estimate of the covariance matrix.

reestimate the regressions using the Cochrane–Orcutt [see Maddala (1977)] procedure. The results from the Cochrane–Orcutt regressions are reported in table 5.

These estimates also reject the null hypothesis that $\beta_2 = 0$; the t-statistics for the β_2 estimates are slightly larger that their counterparts in table 4. This illustrates that the bias in the standard error of the β_2 estimates arising from the slight first-order autocorrelation of the OLS residuals is upward (in the usual case of positively correlated residuals, the bias is often downward). The estimates of β_2 are also very similar to those reported in table 4.

The estimates of β_1 are again insignificant but are all larger than the estimates in table 4. Finally, the estimates of β_0 (obtained by transforming the Cochrane–Orcutt residuals back to the original coordinate system) are all insignificant and about two-thirds the value of their counterparts in table 4. The adjusted R^2s in table 5 are not strictly comparable to those in table 4. The adjusted R^2s in table 5 decline monotonically from 0.569 for the six-month Treasury bills to 0.449 for the twelve-month bills. The Durbin–Watson statistics indicate that the Cochrane–Orcutt procedure is successful in purging the residuals of first-order autocorrelation.

5. Conclusion

We derive an alternative general equilibrium model of the term structure of interest rates within the Cox, Ingersoll, and Ross (1985a) framework. Because this model, which we call the DSR (double square root) model, has the unique feature of implying that yields are nonlinear in the risk-free rate, it is able to generalize many of the empirically relevant features of the SR (square root) model. In addition, the yield nonlinearity leads to a number of new results about the behavior of the term structure of equilibrium interest rates.

The empirical results appear to suggest that the DSR model has incremental explanatory power. Using a GMM procedure, we estimate the parameters of the SR and DSR models and show that the DSR model is more successful in capturing the level and variation of six- to twelve-month Treasury bill yields during the 1964–1986 period. In addition, regression tests suggest that yields are nonlinearly related to the risk-free rate, as the model implies.

These theoretical and empirical results suggest that term structure models that allow yield nonlinearity can provide additional insights and explanatory power for the behavior of equilibrium interest rates. However, the bias in both the SR and DSR models shows that the actual pricing of even intermediate-term discount bonds may be more complex than can be accommodated within the context of a single-state-variable model. Future work could explore discount bond pricing when yields are nonlinear functions of several state variables such as the real rate and the price level.

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