Product Quality and Consumer Search *

José L. Moraga-González[†]

Yajie Sun[‡]

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Abstract

We study the provision of quality in a consumer search market for differentiated products. A raise in quality improves the distribution of match utilities offered by a firm. We show that higher search costs may lead to less investment in quality and, correspondingly, the equilibrium price may decrease in search costs. We provide conditions under which the market may under- or over-supply quality from a social welfare viewpoint. When the market provides an insufficient (excessive) amount of quality, consumers search too little (much).

Keywords: quality investment, sequential search, super- and sub-modular match value distributions, efficiency.

JEL Classification: D43, D83, L13

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[†]Address for correspondence: Vrije Universiteit Amsterdam, Department of Economics, De Boelelaan 1105, 1081HV Amsterdam, The Netherlands. E-mail: `j.l.moragagonzalez@vu.nl`. Moraga is also affiliated with the University of Groningen, the Tinbergen Institute, the CEPR, CESifo, and the PPSRC Center (IESE, Barcelona).

[‡]Vrije Universiteit Amsterdam and Tinbergen Institute. E-mail: `yajie.sun24@gmail.com`.

1 Introduction

Markets in which consumers have to incur search costs to inspect products are ubiquitous. While the literature has paid a lot of attention to the influence of search costs on pricing and concluded that in general search costs confer market power to the firms, much less is known about how frictions affect product features, in particular, product quality.[1] It is plausible that the implied market power of higher search costs gives firms incentives to invest in quality. At the same time, the fact that product quality is costly to observe in a search market may imply that the incentives to invest in quality weaken as search costs increase. This paper aims at studying the private and social incentives to provide quality in a consumer search market. To this ejurre nd, we investigate a version of Wolinsky's (1986) work-horse model in which infinitely many firms selling differentiated products compete by setting prices and investing in quality, while consumers search sequentially until they find satisfactory products. A raise in quality increases the entire distribution of match utilities offered by a firm (in the sense of first-order stochastic dominance), and therefore consumers who pay it a visit are more likely to be satisfied, stop search and buy its product. Investing in quality is of course costly, which generates a trade-off for the firms.

Our first results concern the existence, uniqueness and characterization of a symmetric equilibrium in pure strategies. For fixed firms' strategies, consumer optimal search is characterized by a stationary reservation value that decreases in search cost and increases in firms' investment in quality. Hence, higher quality induces more search. We provide sufficient conditions under which a market equilibrium, that is, a stopping rule for the consumers and a price-investment decision for the firms that are consistent with one another, exists and is unique. These conditions turn out to be related to the notions of *super-* and *submodularity* of the distribution of match utilities, which capture the increasing or decreasing behavior of the density function with respect to quality investment.[2] We provide a variety of examples based on uniform and exponential distributions for which a market equilibrium exists, is unique and can easily be characterized.

We then examine how price and quality investment depend on search costs. To the best of

[1]For a recent survey of firm pricing with consumer search, see Anderson and Renault (2018).

[2]A differentiable λ-parametrized distribution function $F(x;\lambda)$ is said to be supermodular (submodular) if $\partial^2 F/\partial x \partial\lambda > (<) 0$. The concept of supermodularity (submodularity) is the same as increasing (decreasing) differences, that is, $F(x';\lambda') - F(x;\lambda') > (<) F(x';\lambda) - F(x;\lambda)$, $\forall x' > x, \lambda' > \lambda$. As it also turns out, these notions also happen to be key to the understanding of the adequacy of the investment in market equilibrium from the point of view of social welfare (details later).

our knowledge, Ershov (2018) is the only empirical paper measuring the effect of lower search costs on quality. Using data from the Google Play store, Ershov argues that a re-categorization of part of the store that took place in 2014 reduced search costs for game apps and shows empirically that this led to lower quality apps. Our model provides a result consistent with Ershov's finding if the match utility distribution is supermodular and has an increasing density. In such a case, higher search costs lead to more investment in quality. The explanation for this result is as follows. The incentives of a firm to invest in quality are governed by the market value of the implied increase in the acceptance probability of the consumers who pay a visit to the firm. In different words, the gains to a firm from investing in quality relate to the value of an increase in the number of consumers who, after paying a visit to the firm, choose to stop search and buy the product of the firm. An increase in search costs gives consumers incentives to accept products that are less satisfactory, which is reflected in a decrease in consumers' reservation value. When the match value distribution is supermodular and has an increasing density, the decrease in the consumers' reservation value makes both the margin per visitor and the change in the acceptance probability larger. Together, these two effects increase the incentives to invest in quality. For submodular distributions for which the density is not too increasing, the marginal gains from an investment in quality increase in consumers' reservation value and therefore we find the opposite result.

The equilibrium price depends both on the reservation value of consumers and quality investment. The lower the reservation value, the higher the price. Whether the price increases or decreases in quality investment depends again on the way quality affects the density of match values. This is because higher quality may be accompanied by a decrease or an increase in the dispersion of consumer valuations, thereby affecting competition in the market. For distributions satisfying the increasing hazard-ratio ordering (details below), higher quality translates into more dispersed consumer match values and hence into higher prices. Therefore, when search costs go up the equilibrium price also increases (provided that the distribution of match values is supermodular, increasing and has the increasing hazard-ratio property). Otherwise, with submodular distributions a raise in search cost results in less quality supplied in the market and this may result in lower prices. We provide an example in which match utilities are distributed according to the exponential distribution and show that both the equilibrium price and quality decrease as search costs go up.

We finally study how the private incentives to invest in quality fare from a social welfare perspective. Conditional on a given quality investment, consumer search is socially optimal.

Because the equilibrium price does not matter, this implies that the only source of inefficiency in the search market stems from the inadequacy of the incentives to invest in quality. We provide a necessary and sufficient condition for over-investment in quality. Intuitively, excessive investment occurs when the value to a firm of an increase in the number of consumers who stop and buy the product of the firm because it offers more quality is larger than the social value. The private value of a marginal increase in quality can be expressed as the incremental rise in the buying probability per marginal consumer, while the social value equals the aggregate marginal increase in the buying probability for all consumers who buy in the market. This insight points towards a sufficient condition for over-investment: we demonstrate that for distributions whose *mean residual life*, a notion that captures the difference between the expected match value of the consumers who buy and the reservation value (details later), decreases in quality investment, the market over-provides quality. More generally, with supermodular match value distributions with increasing density, we show that the market over-supplies quality relative to the social optimum. In the opposite case in which the match value distribution is submodular and has a decreasing density, the market under-supplies quality.

Related literature

The relationship between the incentives to invest in quality and market power has been of interest in economics at least since Schumpeter (1950). In a seminal contribution, Spence (1975) showed that a monopolist may under- or over-invest compared to the social planner. Our main contribution is to study the welfare aspects of quality provision in a competitive environment with search costs. Because search costs lead to monopolistic competition (Wolinsky, 1986), some of the insights of Spence resonate in our setting. For example, like in Spence's article, we also obtain that the market may provide excessive or insufficient quality. However, a crucial distinction is that while the monopoly price is pivotal to the determination of the efficiency of quality investment, in our model with frictions the equilibrium price does not matter and what really plays a role is the reservation value of consumers. As a matter of fact, we show in Section 8 that if the market was taken over by a monopolist, then quality provision would be optimal in our setting.

As far as we know, work on the efficiency of quality provision in consumer search markets is scant.[3] The first paper is by Wolinsky (2005). There are two important differences between

[3] However, there is a related stream of early work that was mainly motivated by the question whether professionals should be allowed to advertise prices and, in particular, how this price advertising would affect quality provision. Chan and Leland (1982) study a model with all-or-nothing search in which firms are Stackelberg

Wolinsky's setup and ours. The first is that he assumes that an individual firm has to invest in quality every time a consumer pays a visit to the premises of the firm. His model thus better applies to sellers that invest in *service* quality and therefore the notion of quality is short-run in his setup. In our model, by contrast, an individual seller invests once and for all in quality, which corresponds to the more traditional view of quality as a long-run variable. The second main difference is that in Wolinsky's model quality enters additively in the consumers' utility function. We show in Section 7 that with additive quality the market provision of quality is efficient in our setting. Wolinsky, instead, finds that sellers invest too little in service quality, while buyers conduct too much search. The difference is due to quality being short-run in his paper. A second, more recent, paper studying the efficiency of quality provision in a consumer search market is by Chen *et al.* (2019). The main difference between their paper and ours is that Chen *et al.* consider the case of experience goods, that is, products whose quality can only be ascertained after consumption. They find that equilibrium investment in product quality is insufficient (excessive) when search cost is low (high).

Another recent paper on the theme is Fishman and Levy (2015), which presents a model in which firms either sell a product of high- or of low-quality. Also in a framework where quality enters the match value distribution additively, Fishman and Levy study the effects of higher search costs on investment. They find that the effect of higher search costs on quality depend on the initial market distribution of high- and low-quality firms. If the share of high-quality firms is initially low, then higher search costs result in more investment in quality. Fishman and Levy do not study the efficiency of the market equilibrium.

A few papers in the consumer search literature have studied other efficiency aspects of the market equilibrium. Anderson and Renault (1999) study the efficiency of entry; they find that the market equilibrium always provides an excessive number of firms. In a related paper, Chen and Zhang (2015) finds that entry is excessive from a welfare viewpoint when the costs of entering the market are low enough, while for high entry costs entry is deficient for consumer welfare. Finally, Larson (2013) studies the efficiency of the market provision of product design in

leaders vis-á-vis consumers. As in Salop and Stiglitz (1977), there are two types of consumers and in market equilibrium there is either one or two price-quality combinations. They show that allowing for price advertising improves consumer welfare. They extend their model to the case of sequential search in Chan and Leland (1986) and derive conditions under which the same result arises. Rogerson (1988) assumes a sequential search protocol and heterogeneous consumers both in search costs and marginal des-utility of price. He derives an equilibrium with a continuum of price-quality combinations and shows that allowing for price advertising, which activates the role of prices as signals of quality, is shown to improve consumer welfare. A contrasting result is derived by Dranove and Satterthwaite (1992) in a model in which prices and qualities are observed with noise. Random variables are normally distributed and the equilibrium quality turns out to be independent of search costs.

a model similar to Wolinsky (2005). As in Wolinsky's paper, the market and the planner choose to provide the same design conditional on search, so the equilibrium is constraint-efficient.

More generally, our paper adds to a growing literature allowing for a richer choice of firm strategies in consumer search models. In Bar-Isaac *et al.* (2012) firms not only choose the price but also the design of their products. Haan and Moraga-González (2011) present a persuasive advertising game where firms gain prominence by investing in advertising. Moraga-González and Petrikaitė (2013) study firms incentives to merge and the aggregate implications of mergers. Rhodes and Zhou (2015) also study firms incentives to merge and retail various products but in a setting where consumers buy multiple products.

The paper is organized as follows. We introduce the model in section 2. Section 3 is dedicated to the characterisation of the symmetric Nash equilibrium in pure strategies. In this section, we also provide conditions under which a market equilibrium exists and is unique. Section 4 examines the effect of higher search costs on the equilibrium price and quality investment. Section 5 provides illustrative examples. In particular, for a case in which match values follow an exponential distribution, we show that both price and quality decrease in search costs. Section 6 presents the characterisation of the social optimum. In this section, we also provide conditions under which the market over- or under-supplies quality. Section 7 provides a discussion of the main results and some concluding remarks. To ease the reading, all the proofs are relegated to an Appendix.

2 Model

The market has a unit mass of consumers and a unit mass of sellers. Without loss of generality, the marginal cost of production is normalised to zero. In the tradition started by Wolinsky (1986) and followed by many recent consumer search models, the products are horizontally differentiated and consumers search sequentially to find a satisfactory match. The novelty of our model is that firms can make costly efforts to improve their products; we refer to this effort as *investment in quality*.

Consumers are initially imperfectly informed about their fit with the products of the firms and their prices, but they discover these features by paying visits to the firms. We refer to this activity as *search*. Each time a consumer visits a firm, she incurs a search cost, denoted by c. The purpose of search is to inspect the products of the firms and see how well they fit the consumer needs and at which price they sell. In line with the literature, we assume that the search cost is low enough so that consumers find it worthwhile to search; moreover, we assume that consumers hold at all times correct (passive) beliefs about the equilibrium investment in quality and the equilibrium price.[4]

Let p_i be the price charged by firm i. A consumer m who buys product i gets a utility equal to

$$u_i^m = \varepsilon_i^m - p_i,$$

where ε_i^m represents the value of the match between consumer m and product i. We assume that the match values offered by a firm i are distributed on a set $[\underline{\varepsilon}, \overline{\varepsilon}]$ according to a continuous and twice differentiable distribution $F(\varepsilon; \lambda_i)$, with density $f(\varepsilon; \lambda_i)$. The variable $\lambda_i \geq 0$ is the choice of firm i and represents the firm's investment in quality: an increase in λ_i increases the match value distribution in the sense of *first-order stochastic dominance*. To put it in mathematical terms, we assume that

$$\frac{\partial(1 - F(\varepsilon, \lambda_i))}{\partial \lambda_i} \geq 0 \tag{1}$$

and finite for all ε (and strictly positive for some ε's). The lower and upper bounds of the support of ε may or may not depend on λ_i. We restrict the lower bound to be sufficiently low so that the market equilibrium is interior.

Investment in quality is costly. Let $K(\lambda_i)$ represent the cost of investing λ_i; we assume that K is increasing and convex. The case of $\lambda_i = 0$ represents the baseline case of no investment, and we assume that $K(0) = K'(0) = 0$.

Interaction in the market is as follows. Firms simultaneously choose their prices and quality investments. Then, without observing prices, investments and match utilities, consumers search sequentially in the market until they find a satisfactory product. We focus on symmetric pure-strategy equilibrium, that is, an equilibrium in which all the firms charge the same price and make the same investment to improve their products.

Before moving to the equilibrium analysis, we give a couple of examples that are captured by our general modelling of quality investments. The simplest example is the *additive* case (cf. Wolinsky, 2005); in such a case, an investment in quality λ_i increases consumer m's utility from buying product i from $\varepsilon_i^m - p_i$ to $\lambda_i + \varepsilon_i^m - p_i$. In our formulation, this example is captured by the distribution of match values $F(\varepsilon - \lambda_i)$. Notice that in this additive example the utility increases by the same amount no matter the match value. An alternative example

[4] As Bar-Isaac *et al.* (2012) and Larson (2013) argue, in the absence of common factors influencing firms' decisions, with (infinitely many) firms that pick price and quality independently, it is reasonable to expect that a consumer cannot infer much about the deals available at other sellers upon observing a deviation at one of the firms.

is the *multiplicative* case; in such a case, an investment in quality λ_i increases consumer m's utility from product i from $\varepsilon_i^m - p_i$ to $(1+\lambda_i)\varepsilon_i^m - p_i$. In our formulation, this is captured by the distribution of match values $F(\frac{\varepsilon}{1+\lambda_i})$. In the multiplicative case, consumers with high initial match values benefit more from the increase in quality than the others.

3 Market equilibrium

In this section, we study the existence and uniqueness of a symmetric equilibrium of the model. In a symmetric equilibrium all sellers choose the same quality level and charge the same price; as a result, the utility distribution is the same across all sellers. Moreover, all consumers search in the same way, holding correct conjectures about prices and match utility distributions.

3.1 Consumer optimal search

Let (p^*, λ^*) be the symmetric firm equilibrium. Because all the firms offer the same utility distribution, we can rely on Kohn and Shavell (1974), who show that the optimal search rule for a consumer who faces a set of independently and identically distributed options with a known distribution is static in nature and has the stationary reservation utility property.

Accordingly, consider the equation

$$h(\varepsilon) \equiv \int_{\varepsilon} (z-\varepsilon) f(z,\lambda^*) dz = c. \qquad (2)$$

The left-hand-side (LHS) of Equation (Eq.) (2) is the equilibrium marginal benefit of search to a consumer who has a match value ε at hand, which is clearly a decreasing function of ε. The right-hand-side (RHS) is the cost of an additional search. If the maximum of the LHS is smaller than the search cost, then it is not worth for a consumer to enter the market. Otherwise, the consumer enters the market and uses the unique solution to Eq. (2) as her reservation value, which we denote $\hat{\varepsilon}$.

Notice that the reservation value $\hat{\varepsilon}$ is a function of the search cost c and the equilibrium investment in quality λ^*. As a matter of fact, it is easy to show that (see proof of Proposition 1), for a given search cost c, $\hat{\varepsilon}$ is an increasing function of λ^*; this means that when the market supplies better products in the sense of FOSD, consumers are prepared to search more thoroughly in the market until they find a satisfactory match. For a fixed λ^*, the reservation value $\hat{\varepsilon}$, as usual, decreases in the search cost c, revealing that consumer curtail their search effort when the search cost goes up.

3.2 Firm pricing and investment

We now characterise the symmetric equilibrium price and investment level (p^*, λ^*). To do this, we write out the payoff to a firm, say i, that deviates from the proposed equilibrium by charging a price p_i and investing an amount λ_i, taking as given other firms' equilibrium decisions and consumers' search behaviour. After this, we compute the first order conditions (FOCs), apply symmetry and study the existence of the symmetric (price and investment) equilibrium.

Given other firms' decisions and consumers search behaviour, the payoff to a deviant firm charging a price $p_i \neq p^*$, and investing an amount $\lambda_i \neq \lambda^*$ is equal to:

$$\pi_i(p_i, \lambda_i; p^*, \lambda^*) = \frac{p_i[1 - F(\hat{\varepsilon} - p^* + p_i; \lambda_i)]}{1 - F(\hat{\varepsilon}; \lambda^*)} - K(\lambda_i). \tag{3}$$

This expression for the profits of a firm should be understood as follows. The per consumer revenue of the deviant firm is p_i; the number of consumers who visit the deviant firm is $1/(1 - F(\hat{\varepsilon}; \lambda^*))$; the probability that one of these visitors chooses to stop searching and buys the product of the deviant firm is the probability of the event $\varepsilon_i - p_i \geq \hat{\varepsilon} - p^*$, which gives $1 - F(\hat{\varepsilon} - p^* + p_i; \lambda_i)$. The costs of investing λ_i enter the profits formula negatively.

Taking the first-order derivatives of the profits function with respect to λ_i and p_i and applying symmetry ($p_i = p^*$, $\lambda_i = \lambda^*$) gives the following FOCs:

$$p^* = \frac{1 - F(\hat{\varepsilon}; \lambda^*)}{f(\hat{\varepsilon}; \lambda^*)} \tag{4}$$

$$p^* \frac{\frac{\partial(1 - F(\hat{\varepsilon}; \lambda^*))}{\partial \lambda}}{1 - F(\hat{\varepsilon}; \lambda^*)} - K'(\lambda^*) = 0. \tag{5}$$

Plugging p^* into the second expression, we can rewrite the FOCs (4)-(5) as follows:

$$p^* = \frac{1 - F(\hat{\varepsilon}, \lambda^*)}{f(\hat{\varepsilon}, \lambda^*)} \tag{6}$$

$$0 = \frac{\frac{\partial(1 - F(\hat{\varepsilon}, \lambda^*))}{\partial \lambda}}{f(\hat{\varepsilon}, \lambda^*)} - K'(\lambda^*). \tag{7}$$

For a given $\hat{\varepsilon}$, if a symmetric equilibrium exists, all sellers choose p^* and λ^* satisfying (6) and (7) as their optimal strategy. We now provide conditions under which a firm equilibrium exists and is unique.

Lemma 1 *If the investment cost $K(\lambda)$ is sufficiently convex, f is non-decreasing in ε and $\frac{\partial(1 - F(\hat{\varepsilon}, 0))}{\partial \lambda} > 0$, then there exists a unique symmetric firm equilibrium. The firm equilibrium is the pair (p^*, λ^*) that solves (6) and (7).*

Proof. See the Appendix. ■

Note that the assumption in Lemma 1 that $\frac{\partial(1-F(\hat{\varepsilon},0))}{\partial\lambda} > 0$ is innocuous when investment shifts the entire distribution downwards, which will be the typical case in the examples used in the remaining of the paper. However, more in general, the FOSD assumption only implies that the term $\frac{\partial(1-F(\hat{\varepsilon},0))}{\partial\lambda}$ will be strictly positive for some ε's, and therefore not necessarily for $\hat{\varepsilon}$; that is the reason for which we make this assumption explicit in the Lemma. The assumption that f is non-decreasing ensures that the payoff is strictly concave in p_i but it is obvious that f not too decreasing may be sufficient.

3.3 Market equilibrium

In the above two sections, we have seen how consumers search optimally when they expect a price p^* and a level of investment λ^*; likewise, we have seen how firms price and invest in quality when they expect consumers to search using a reservation value $\hat{\varepsilon}$. A market equilibrium exists if there is a triplet $(\hat{\varepsilon}, p^*, \lambda^*)$ that simultaneously solves the consumers' and the sellers' problems. Because the price p^* is uniquely pinned down by $\hat{\varepsilon}$ and λ^* using Eq. (6), a market equilibrium exists if the following system of equations has a solution in $(\hat{\varepsilon}, \lambda^*)$:

$$h(\hat{\varepsilon}, \lambda^*) \equiv \int_{\hat{\varepsilon}} (z - \hat{\varepsilon}) f(z, \lambda^*) dz - c = 0 \tag{8}$$

$$l(\hat{\varepsilon}, \lambda^*) \equiv \frac{\frac{\partial(1-F(\hat{\varepsilon},\lambda^*))}{\partial\lambda}}{f(\hat{\varepsilon}, \lambda^*)} - K'(\lambda^*) = 0 \tag{9}$$

This system combines the consumers' search rule (Eq. (8)) and the sellers' profit maximization problem (Eq. (9)).

Eq. (8), which represents the consumers' stopping rule, is well known in the literature. This equation defines an implicit relation between the reservation value and investment. As we show below, this relation is monotonically increasing, which reflects the idea that there is more search in markets in which the products are of higher quality. Eq. (9) also defines a relationship between the reservation value and investment but it turns out that this relationship can be increasing or decreasing, depending on the properties of the density function of match values. When the relation is increasing, firms incentives to invest increase if consumers are prepared to search more; otherwise, more search gives firms less incentives to put effort in improving their products.

Our next result provides sufficient conditions under which a market equilibrium exists and is unique. We provide two sets of conditions, depending on whether Eq. (9) defines an increasing or a decreasing relationship between the reservation value and investment. Later we provide some examples illustrating these two cases.

Proposition 1 *If $K(\lambda)$ is sufficiently convex, f is non-decreasing in ε and F satisfies either of the following conditions:*

(a) $\frac{\partial f}{\partial \lambda} f + \frac{\partial(1-F)}{\partial \lambda}\frac{\partial f}{\partial \varepsilon} \geq 0$ *and* $\frac{\partial(1-F(\bar{\varepsilon},0))}{\partial \lambda} = 0$

(b) $\frac{\partial f}{\partial \lambda} f + \frac{\partial(1-F)}{\partial \lambda}\frac{\partial f}{\partial \varepsilon} < 0$ *and* $\frac{\partial(1-F(\underline{\varepsilon},0))}{\partial \lambda} = 0,$

then a unique market equilibrium exists. In equilibrium, the equilibrium price is given by (6), while consumers reservation value $\hat{\varepsilon}$ and firms quality investment λ^* solve (8) and (9).

Proof. See the Appendix. ∎

A discussion on the conditions in Proposition 1 is now in order. In general, the conditions in Proposition 1(a) hold for any supermodular distribution with non-decreasing density, i.e. distributions for which $\frac{\partial f}{\partial \lambda} > 0$ and $\frac{\partial f}{\partial \varepsilon} \geq 0$, and with an upper bound independent of λ. One example is the family of distributions $F(\varepsilon,\lambda) = 1 - \left(1 - \frac{\varepsilon-\lambda}{1-\lambda}\right)^{\alpha}$, $\alpha \leq 1$, defined on $[\lambda, 1]$, which belongs to the family of Kumaraswamy distributions and includes the uniform distribution $F(\varepsilon,\lambda) = \frac{\varepsilon-\lambda}{1-\lambda}$.[5] As mentioned above, the requirement that the density is non-decreasing in ε is quite strong. Another example that satisfies the conditions in Proposition 1(a) is the family $F(\varepsilon,\lambda) = \frac{(\varepsilon-\lambda)(1-\lambda\varepsilon)}{(1-\lambda)^2}$, defined on $[\lambda, 1]$. Although this family has a decreasing density in ε, it verifies the conditions.

The conditions in Proposition 1(b) will be satisfied for submodular distributions with non-decreasing densities provided that the lower bound does not change with λ. One example is the family of distributions $F(\varepsilon,\lambda) = 1 - \left(1 - \frac{\varepsilon}{1+\lambda}\right)^{\alpha}$, $\alpha \leq 1$, defined on $[0, 1+\lambda]$, which includes the uniform distribution $F(\varepsilon,\lambda) = \frac{\varepsilon}{1+\lambda}$. The power distribution $F = \left(\frac{\varepsilon}{1+\lambda}\right)^{\alpha}$, $\alpha \geq 1$, defined on $[0, 1+\lambda]$ also satisfies the conditions in Proposition 1(b). As mentioned above, these requirements are quite strong. The exponential distribution $F = 1 - e^{-\frac{\varepsilon}{1+\lambda}}$, defined on $[0, \infty]$ satisfies the conditions in Proposition 1(b) despite not being submodular and having a decreasing density. In what follows, we shall assume that an equilibrium exists and is unique.

[5] The Kumaraswamy (1980) distribution function is

$$F(\varepsilon,\lambda) = 1 - \left[1 - \left(\frac{\varepsilon-\lambda}{1-\lambda}\right)^{a}\right]^{b}, \ \varepsilon \in [\lambda, 1], \ a, b > 0.$$

The Kumaraswamy distribution is often used as a substitute for the beta distribution (see Ding and Wolfstetter, 2011).

4 Higher search costs

We now ask how the market equilibrium is affected by higher search costs. Interestingly, investment can increase or decrease, and the equilibrium price, in contrast to the conventional result in the literature, need not increase in search cost.

Before presenting our result, we introduce the hazard rate ordering of random variables.

Definition 1 *The random variable ε' with distribution $F(\varepsilon', \lambda')$ is said to dominate the random variable ε with distribution $F(\varepsilon, \lambda)$ in the sense of the hazard rate if and only if*

$$\frac{f(z, \lambda')}{1 - F(z, \lambda')} \leq \frac{f(z, \lambda)}{1 - F(z, \lambda)}, \text{ for all } z \text{ and } \lambda' > \lambda.$$

Note that the hazard rate ordering implies the first-order stochastic dominance ordering (not vice versa).

A higher quality raises consumer valuations. However, *ceteris paribus*, a higher quality does not necessarily imply a higher price. This is because a higher quality may also decrease the dispersion of consumer valuations, thereby increasing market competitiveness. Because the equilibrium price is equal to the inverse of the hazard rate, the hazard rate ordering of random variables is useful to determine how the equilibrium price depends on investment keeping everything else constant. Specifically, for distributions that satisfy the hazard rate ordering, a higher quality translates into a higher price. We use this in our next result.

Proposition 2 *As search cost s increases, the equilibrium reservation value $\hat{\varepsilon}$ decreases. Moreover:*

(a) Under the conditions in Proposition 1(a), the equilibrium investment level λ^* increases with the search cost, while the behaviour of the equilibrium price is in principle ambiguous. For distributions satisfying the hazard-ratio ordering, the equilibrium price increases in search cost.

(b) Under the conditions in Proposition 1(b), the equilibrium investment level λ^* decreases as search cost increases, while the effect of higher costs on the equilibrium price can be positive or negative. For match values following the exponential distribution $F = 1 - e^{-\frac{\varepsilon}{1+\lambda}}$, defined on $[0, \infty]$, the equilibrium price decreases in search cost.

Proof. See the Appendix. ■

The intuition behind this result is as follows. Let us start by explaining how the incentives to invest in quality, which are given by the LHS of Eq. (9), vary with search costs. Conditional on a consumer visiting a firm, the derivative $\partial(1-F(\hat{\varepsilon},\lambda))/\partial\lambda$ represents the marginal increase in her buying probability generated by an rise in quality. The density $f(\hat{\varepsilon},\lambda)$ represents the number of consumers at the margin. The incentives of a firm to invest in quality are thus governed by the incremental rise in the acceptance probability per consumer at the margin. An increase in search costs gives consumers incentives to accept products that are less satisfactory, which is reflected in a decrease in consumers' reservation value. When the match value distribution is supermodular and has an increasing density, the fall in the consumers' reservation value makes the increase in the acceptance probability bigger and the number of consumers at the margin lower. Together, these two effects increase the incentives to invest in quality. By contrast, for submodular distributions for which the density is not too increasing, the marginal gains from an investment in quality increase in consumers' reservation value and therefore higher search costs result in a lower quality investment.

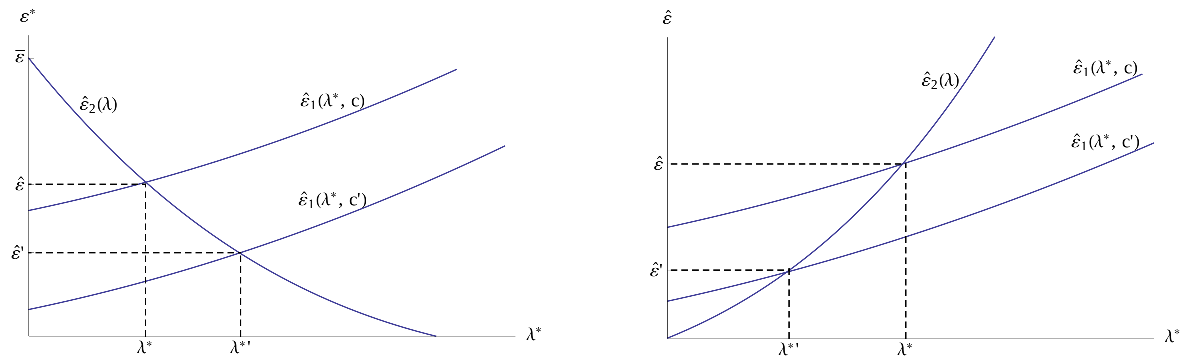
Let us continue now by explaining the effect of higher search costs on the equilibrium price. The equilibrium price depends both on the reservation value of consumers and quality in- vestment. An increase in the search cost lowers the reservation value and this tends to increase the equilibrium price. However, as mentioned above, depending on the behaviour of the properties of the match value distribution, an increase in the search cost may result in more or in less investment in quality. When the match value distribution is supermodular and has an increasing density, in particular, it results in more investment in quality. With distributions satisfying the hazard-ratio ordering, more investment in quality also tends to raise the equilibrium price. Together increasing density and the hazard-ratio ordering imply that the distribution of match values becomes more dispersed after investment,[6] which weakens competition and tends to increase the price. In total, thus, higher search costs lead to more quality and higher prices.

Proposition (2) is illustrated in Figure 1. An increase in search costs shifts the $\varepsilon_1^*(\lambda)$ schedule downwards. If the schedule $\varepsilon_2^*(\lambda)$ is decreasing as in Figure 1(a), this results in a lower reservation value and a higher investment level. A lower reservation value pushes the price up, while the effect of a higher investment level on the price may be positive or negative. With distributions satisfying the hazard-ratio ordering, the price increases in search costs.

If the schedule $\varepsilon_2^*(\lambda)$ is increasing as in Figure 1(b), an increase in the search cost results in a lower reservation value and a lower investment level. The effect on price is again ambiguous. A higher search cost lowers the reservation value and this results in a higher price. At the same

[6]See Theorem 3.B.20 in Shaked and Shanthikumar (2007).

time, a higher search cost decreases investment and this, again, can push the price up or down. For the case in which the match value distribution is exponential, the equilibrium price will decrease in search costs (see Example 3 below for more details).



(a) Investment decreases in c

(b) Investment increases in c

Figure 1: Equilibrium investment and search cost

5 Examples

In this section we provide a couple of examples to illustrate the results obtained so far. The examples are based on uniform and exponential distributions. The first two examples satisfy the conditions in Proposition 1(a). The last two examples those in Proposition 1(b).

We start with a uniform example that satisfies the conditions in Proposition 1(a) and has weakly increasing hazard rate. In this example, price and investment both strictly increase in search costs.

Example 1 ***Match values uniformly distributed on*** $[\lambda, 1]$
Consider the case in which match values follow the uniform distribution

$$F(\varepsilon, \lambda) = \frac{\varepsilon - \lambda}{1 - \lambda},$$

with support on $[\lambda, 1]$*, where* $0 \leq \lambda < 1$*. The density function is* $f(\varepsilon, \lambda) = \frac{1}{1-\lambda}$*, which is non-decreasing in* ε *and increasing in* λ *(so* F *is supermodular). The case of* $\lambda = 0$ *is the case of the standard uniform distribution. Note that* $\frac{\partial(1-F)}{\partial\lambda} = \frac{1-\varepsilon}{(1-\lambda)^2} \geq 0$*, therefore an increase in* λ *leads to increase in match values in the FOSD sense. Moreover, the conditions in Proposition 1(a) hold for this case:*

$$\frac{\partial f}{\partial \lambda} f + \frac{\partial(1-F)}{\partial \lambda}\frac{\partial f}{\partial \varepsilon} = \frac{1}{(1-\lambda)^3} > 0$$

and the solution to $\frac{\partial(1-F(\varepsilon,0))}{\partial\lambda} = 0$ *is* $\varepsilon = 1$*.*

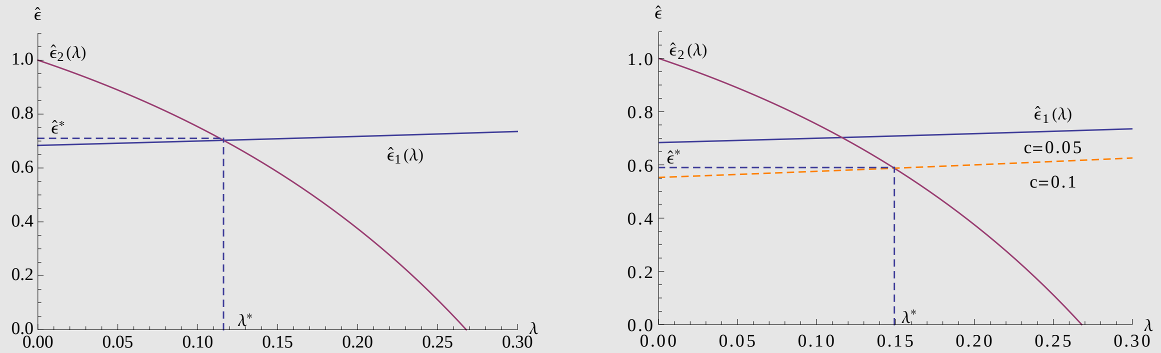
Let the cost of investment in quality function be $K(\lambda) = \frac{\lambda^2}{(1-\lambda)^2}$*, which is increasing and convex.*

The market equilibrium conditions (8)-(9) can be written as:

$$\varepsilon = 1 - \sqrt{2c(1-\lambda)} \tag{10}$$

$$\varepsilon = 1 - \frac{2\lambda}{(1-\lambda)^2} \tag{11}$$

It is easily seen that there exists a unique solution to this system of equations. Inspection of the expression on the RHS of Eq. (10) immediately reveals that it is increasing in λ*, going from* $1-\sqrt{2c}$ *to 1 as* λ *increases from 0 to 1. Meanwhile, the expression on the RHS of Eq. (11) has derivative with respect to* λ *equal to* $-\frac{2(1+\lambda)}{(1-\lambda)^3} < 0$*, so it is decreasing in* λ*, going from 1 to* $-\infty$*. Because both expressions are continuous, they must cross once and only once, therefore guaranteeing a unique solution. We illustrate these observations in Figure 2. We plot Eqs. (10) and (11) in Figure 2(a) for c = 0.05. The effect of an increase in search costs can be seen in Figure 2(b), where we also plot Eq. (10) for c = 0.1. The equilibrium reservation value decreases while the investment level increases in search costs.*



(a) Equilibrium

(b) Higher seach cost

Figure 2: Uniform distribution on $[\lambda, 1]$

The equilibrium price is $p^* = 1 - \hat{\varepsilon} = \frac{2\lambda^*}{(1-\lambda^*)^2}$*, which is increasing in quality investment* λ^* *and therefore in search costs. Despite the quality increase, consumer surplus, which is equal to* $CS = \hat{\varepsilon} - p^* = 2\hat{\varepsilon} - 1$*, decreases in search costs. Consumers participate in the market provided that* $\lambda^* \leq 3 - 2\sqrt{2}$*, or if search costs are lower than, approximately,* 0.15*. The profits of a firm are:*

$$\pi = p^* - K(\lambda^*) = \frac{\lambda^*(2-\lambda^*)}{(1-\lambda^*)^2},$$

which are non-negative for all $\lambda^* \in (0,1)$*, increase in* λ^* *and therefore in search costs. Welfare, defined as the sum of consumer surplus and firms profits, equals* $W = (1-4\lambda^*)/(1-\lambda^*)^2$ *and is also decreasing in search costs.* $\square$

We continue with an exponential example that satisfies the conditions in Proposition 1(a); in this case, however, the price and investment are constant in search costs.

Example 2 ***Match values exponentially distributed on*** $[\lambda, \infty)$
Consider next the case in which match values are exponentially distributed on the set $[\lambda, \infty)$. *The distribution function is*

$$F(\varepsilon, \lambda) = 1 - e^{-(\varepsilon - \lambda)}.$$

Note that $\frac{\partial(1-F)}{\partial\lambda} = e^{-(\varepsilon-\lambda)} > 0$ *so higher* λ *means higher match values in the FOSD sense. The density function is* $f(\varepsilon, \lambda) = e^{-(\varepsilon-\lambda)}$. *This density increases in* λ *(so* F *is supermodular) but decreases in* ε. *Nevertheless, the conditions in Proposition 1(a) hold for this case because*

$$\frac{\partial f}{\partial \lambda} f + \frac{\partial(1-F)}{\partial \lambda} \frac{\partial f}{\partial \varepsilon} = 0$$

and the solution to $\partial(1-F)/\partial\lambda = 0$ *is* $\varepsilon = \infty$. *As in the previous example, let* $K(\lambda) = \frac{\lambda^2}{(1-\lambda)^2}$. *The market equilibrium conditions (8)-(9) can be written as:*

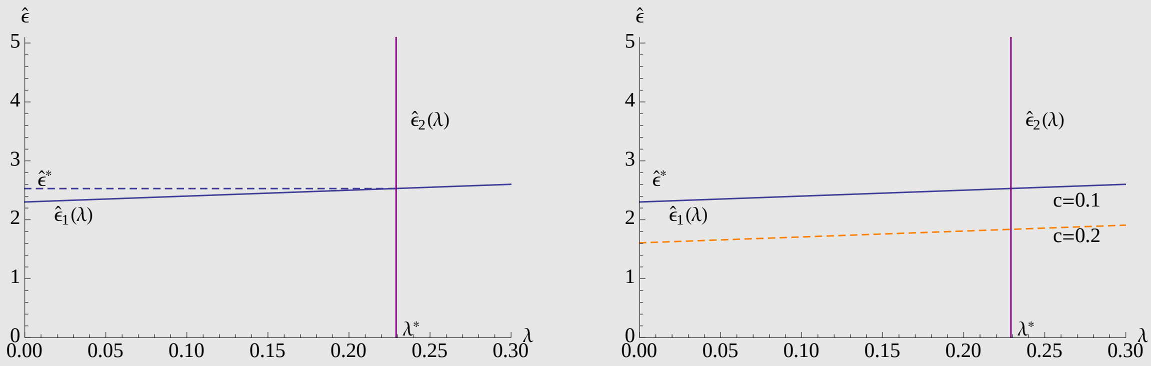
$$\varepsilon = \lambda - \log c \tag{12}$$

$$1 = \frac{2\lambda}{(1-\lambda)^3} \tag{13}$$

Inspection of this system of equations shows it is recursive. Equation (13) pins λ *down directly:* $\lambda^* \simeq 0.23$ *and the reservation values follows after plugging* λ^* *in Eq. (12).*
The equilibrium conditions (12)-(13) are depicted in Figure 3. In Figure 3(a) we have plotted Eq. (12) for a search cost $c = 0.1$. *The vertical line represents Eq. (13). The crossing point between the two lines gives the equilibrium reservation value. In Figure 3(b) we show the effect of an increase in search cost from* $c = 0.1$ *to* $c = 0.2$. *Clearly, a higher search cost decreases the reservation value of consumers but has no bearing on investment. The equilibrium price is* $p^* = 1$, *which happens to be independent of the reservation value and investment.*
The profits of a typical firm are then:

$$\pi = p^* - K(\lambda^*) = 1 - \frac{\lambda^{*2}}{(1-\lambda^*)^2} \simeq 0.91$$

and consumer surplus $CS = \hat{\varepsilon} - p^* = \lambda^* - \log c - 1 \simeq -\log c - 0.77$, *which is positive for sufficiently low search costs.*



(a) Equilibrium

(b) Higher seach cost

Figure 3: Exponential distribution on $[\lambda, \infty)$

We proceed now to an example with uniformly distributed match values that satisfies the conditions in Proposition $1(b)$. In this example, investment decreases in search costs but the price still increases.

Example 3 ***Match values uniformly distributed on*** $[0, 1 + \lambda]$
Consider the case in which match values are distributed uniformly on the set $[0, 1 + \lambda]$*. In this case,*

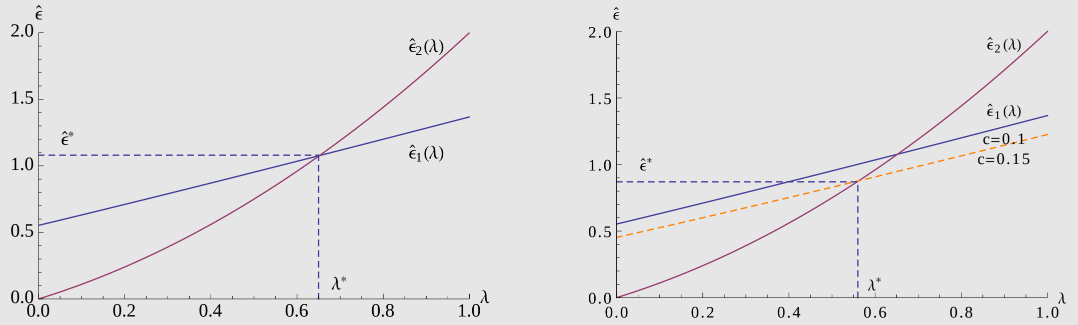
$$F(\varepsilon, \lambda) = \frac{\varepsilon}{1 + \lambda}.$$

The case of $\lambda = 0$ *is again the case of the standard uniform distribution on* $[0, 1]$ *and as* λ *increases the distribution of match values increases in the FOSD sense. The conditions in Proposition 1(b) hold. The corresponding density function is* $f(\varepsilon, \lambda) = 1/(1 + \lambda)$*, which is constant in* ε *and decreasing in* λ *(so* F *is submodular); moreover,* $\partial(1-F)/\partial\lambda = \varepsilon/(1+\lambda)^2$*, which is equal to zero when* $\varepsilon = 0$*.*
Let $K(\lambda) = \frac{\lambda^2}{2}$*, which is increasing and convex. For the case at hand, the market equilibrium equations (8) and (9) can be written as:*

$$\varepsilon = 1 + \lambda - \sqrt{2c(1 + \lambda)} \tag{14}$$

$$\varepsilon = \lambda(1 + \lambda) \tag{15}$$

It is easy to see that there is a unique pair (ε, λ) *that satisfies the above system of equations. Note that the RHS of Eq. (14) takes on value* $1 - \sqrt{2c}$ *when* $\lambda = 0$ *and increases in* λ *with a slope equal to* $1 - \frac{c}{\sqrt{2c(1+\lambda)}}$*. The RHS of (15) takes on value 0 when* $\lambda = 0$ *and increases in* λ *with a rate equal to* $1 + \lambda$*, which is greater than the rate at which the RHS of Eq. (14) rises. We conclude that there is a unique crossing point. We illustrate these observations in Figure 4. We plot Eqs. (14) and (15) in Figure 4(a) for* $c = 0.01$*. The effect of an increase in search costs can be seen in Figure 4(b), where we also plot Eq. (14) for* $c = 0.05$*. Both the equilibrium reservation value and investment level decrease in search cost.*



(a) Low search cost (b) High search cost

Figure 4: Uniform distribution

The equilibrium price, which is given by $p^* = 1 + \lambda^* - \hat{\varepsilon}$ *can be written using (15) as* $p^* = 1 - \lambda^{*2}$*, which clearly increases in search costs. Consumer surplus is* $CS = \hat{\varepsilon} - p^* = \lambda^*(1 + 2\lambda^*) - 1$*, which increases in* λ^* *and therefore decreases in search costs. Consumers enter the market only if* $\lambda^* \geq 1/2$ *so the search cost has to be smaller than 3/16. Equilibrium profits are equal to* $\pi^* = 1 - 3\lambda^2/2$*; to guarantee that firms make positive profits in equilibrium, we need that* $\lambda < \sqrt{2/3}$*, which holds when the search cost is not too low.* □

Finally we present an example based on the exponential distribution for which investment in quality and the equilibrium price decreases in search costs.

Example 4 ***Match values exponentially distributed on*** $[0, \infty)$
Consider now the case in which match values follow an exponential distribution with parameter $\frac{1}{1+\lambda}$ *and support* $[0, \infty)$*. In this case, the distribution is*

$$F(\varepsilon, \lambda) = 1 - e^{-\frac{1}{1+\lambda}\varepsilon}$$

and the density is $f(\varepsilon, \lambda) = \frac{1}{1+\lambda} e^{-\frac{1}{1+\lambda}\varepsilon}$*. The case of* $\lambda = 0$ *is the case of exponential with mean 1. Note that an increase in* λ *shifts the distribution downwards so match values increase in the FOSD sense. The density is decreasing in* ε *and its behavior with respect to* λ *is ambiguous. Nevertheless, it is easy to verify that the conditions in Proposition 1(b) hold for this exponential case. In fact,*

$$\frac{\partial f}{\partial \lambda} f + \frac{\partial(1-F)}{\partial \lambda}\frac{\partial f}{\partial \varepsilon} = -\frac{e^{-\frac{2\varepsilon}{1+\lambda}}}{(1+\lambda)^3} < 0,$$

where we have used the following derivatives:

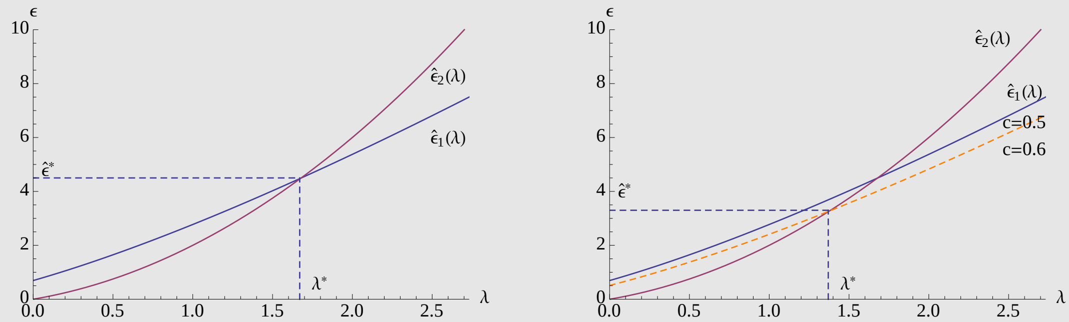
$$\frac{\partial f}{\partial \lambda} = -\frac{e^{-\frac{x}{1+\lambda}}(1+\lambda-\varepsilon)}{(1+\lambda)^3}, \quad \frac{\partial(1-F)}{\partial \lambda} = \frac{\varepsilon e^{-\frac{\varepsilon}{1+\lambda}}}{(1+\lambda)^2}, \text{ and } \frac{\partial f}{\partial \varepsilon} = -\frac{e^{-\frac{\varepsilon}{1+\lambda}}}{(1+\lambda)^2}.$$

Assume that $K(\lambda) = \frac{\lambda^2}{2}$*. After simplification, the market equilibrium equations (8) and (9)* *become:*

$$\varepsilon = (1+\lambda)\ln\left[\frac{1+\lambda}{c}\right] \tag{16}$$

$$\varepsilon = \lambda(1+\lambda) \tag{17}$$

It is easy to verify that the system of Eqs. (16)-(17) has a unique solution. Like before, an increase in search costs shifts Eq. (16) downwards and therefore the equilibrium reservation value and investment level decrease. In this case the equilibrium price is $p^* = 1+\lambda$, *which also decreases as the search cost increases. Equilibrium profits are equal to* $\pi^* = 1+\lambda-\lambda^2/2$; *to guarantee that firms make positive profits in equilibrium, we need that* $\lambda < 1+\sqrt{3}$, *which holds when the search cost is sufficiently high.*
The case at hand is illustrated in Figure 6. We plot Eqs. (16) and (17) in Figure 4(a) for $c = 0.5$. *The effect of an increase in search costs can be seen in Figure 5(b), where we also plot Eq. (16) for* $c = 0.6$. *The equilibrium reservation value, the investment level and the equilibrium price decrease in search cost.*



(a) Equilibrium

(b) The effect of a higher search cost

Figure 5: Exponential distribution

6 Efficiency

In this section, we ask whether the market provides too little or too much quality from a social welfare perspective. To address this question, we derive the social (first-best) optimum and compare it to the market equilibrium of Proposition 1. Social welfare is measured by the sum of expected consumer surplus and expected industry profit. If the investment level is λ, price is p and consumers search with a reservation value $\hat{\varepsilon}$, then the expected consumer surplus, denoted by CS, can be written as:

$$CS(\lambda, \hat{\varepsilon}) = \frac{\int_{\hat{\varepsilon}}(z-p)f(z,\lambda)dz - c}{1 - F(\hat{\varepsilon}, \lambda)}.$$

The profits of a firm are equal to its revenues minus its costs. The revenue is simply the price because the number of consumers per firm is normalized to one and all consumers are served in equilibrium. The cost is just the investment cost because the marginal cost of production is normalized to zero. Because there is a unit mass of sellers, industry profit is equal to:

$$\Pi(\lambda, \hat{\varepsilon}) = p - K(\lambda).$$

Summing consumer surplus and industry profits we obtain an expression for welfare:

$$W(\lambda, \hat{\varepsilon}) = \frac{\int_{\hat{\varepsilon}} z f(z, \lambda) dz - c}{1 - F(\hat{\varepsilon}, \lambda)} - K(\lambda). \tag{18}$$

Notice that in the welfare expression the price p cancels out. This is because the price is just a transfer between consumers and firms and has no bearing on aggregate surplus.

Taking the first order conditions of the social welfare expression in (18) with respect to $\hat{\varepsilon}$ and λ gives:

$$\frac{\partial W}{\partial \hat{\varepsilon}} = -\hat{\varepsilon} f(\hat{\varepsilon}, \lambda)(1 - F(\hat{\varepsilon}, \lambda)) + f(\hat{\varepsilon}, \lambda)\left[\int_{\hat{\varepsilon}} z f(\hat{\varepsilon}, \lambda) dz - c\right] = 0, \tag{19}$$

$$\frac{\partial W}{\partial \lambda} = \frac{[1 - F(\hat{\varepsilon}, \lambda)] \int_{\hat{\varepsilon}} \frac{\partial (1 - F(z, \lambda))}{\partial \lambda} dz - \frac{\partial (1 - F(\hat{\varepsilon}, \lambda))}{\partial \lambda} [\int_{\hat{\varepsilon}} (z - \hat{\varepsilon}) f(z, \lambda) dz - c]}{[1 - F(\hat{\varepsilon}, \lambda)]^2} - K'(\lambda). \tag{20}$$

After rearranging, the FOC (19) can be rewritten as $\int_{\hat{\varepsilon}} (z - \hat{\varepsilon}) f(\hat{\varepsilon}, \lambda) dz - c = 0$, which can be used to simplify (20) and rewrite the first order conditions for social welfare maximization more compactly as:

$$\int_{\hat{\varepsilon}} (z - \hat{\varepsilon}) f(\hat{\varepsilon}, \lambda) dz - s = 0 \tag{21}$$

$$\frac{\int_{\hat{\varepsilon}} \frac{\partial (1 - F(z, \lambda))}{\partial \lambda} dz}{1 - F(\hat{\varepsilon}, \lambda)} - K'(\lambda) = 0 \tag{22}$$

A close inspection of this system of equations reveals one important fact. Comparing (21) to the market search rule (8), we observe that they are exactly identical. This implies that for any exogenously fixed investment level λ, consumer search in the market equilibrium is efficient. Therefore, if the equilibrium has an efficient amount of investment, then search will also be efficient; otherwise, inefficient investment will result in inefficient search.

Let us denote the social optimum that solves Eqs. (21) and (22) by $(\hat{\varepsilon}^o, \lambda^o)$. Providing conditions under which the social welfare function is globally concave is quite hard. We will not pursue this any further. Instead, we now move to study whether the market provides too much or too little incentives to invest in quality.

Proposition 3 *The market (over-) under-provides quality and, consequently, consumers search too (much) little if and only if*

$$\frac{\int_{\hat{\varepsilon}} \frac{\partial(1-F(z,\lambda))}{\partial\lambda} dz}{1-F(\hat{\varepsilon},\lambda)} (<) > \frac{\frac{\partial(1-F(\hat{\varepsilon},\lambda))}{\partial\lambda}}{f(\hat{\varepsilon},\lambda)} \tag{23}$$

Proof. See the Appendix. ■

The condition in Proposition 3 stems from a comparison of the social and the private incentives to invest in quality. The economic interpretation of this condition is as follows. The LHS of Eq. (23) represents the marginal social gains from an investment in quality. The derivative $\partial(1-F(z,\lambda))/\partial\lambda$ captures the increase generated by an rise in quality in the quantity demanded when the reservation value is equal to z. In different words, it measures the increase in the probability that a consumer has a value more than z. Hence, the LHS of Eq. (23) gives the average marginal increase in the quantity of consumers who buy in the market due to an extra unit of quality. On the RHS of Eq. (23), we have the private incentives to invest in quality. The private incentives are given by incremental rise in the stopping probability per consumer at the margin.

An individual firm and the social planner have different valuations for quality increments, thereby creating a source of potential market failure. While the social planner cares about the average increase in the quantity demanded, an individual firm only cares about the marginal increase in the stopping probability per marginal consumer.

6.1 Two sufficient conditions for over-investment in quality

Proposition 3 gives a necessary and sufficient condition under which the market provides too much quality. The condition is however hard to check so in this subsection, we give two propositions providing sufficient conditions for over-investment that are intuitive and easy to verify. In Section 6.2 we provide a sufficient condition for under-investment in quality.

Proposition 4 *If the distribution of match values $F(\varepsilon,\lambda)$ is supermodular and the density weakly increases in ε, then the market over-supplies quality.*

Proof. See the Appendix. ■

The intuition behind this result is as follows. With supermodular distributions, the density of match values increases in quality investment. This implies that the marginal increase in the probability with which a consumer stops searching and buys the product of the firm when quality goes up is larger the lower the match value. As a result, the incremental rise in the stopping probability of the marginal consumer is higher than that of the average consumer. In mathematical terms, with supermodular distributions we have

$$\frac{\partial(1-F(\varepsilon,\lambda)}{\partial\lambda} > \frac{\int_{\varepsilon}^{\bar{\varepsilon}} \frac{\partial(1-F(z,\lambda)}{\partial\lambda} dz}{\int_{\varepsilon}^{\hat{\varepsilon}} dz}.$$

If, in addition, the density is increasing then the frequency of marginal consumers is lower than that of the average consumer who buys, that is

$$f(\varepsilon,\lambda) \leq \frac{1-F(\varepsilon,\lambda)}{\int_{\varepsilon}^{\bar{\varepsilon}} dz}.$$

Together, these two conditions suffice for overinvestment.

We now return to Example 1, for which we know that the conditions in Proposition 4 hold, and verify that firms invest too much from the perspective of social welfare maximisation. As a result, consumers search too much (though given the quality they search efficiently).

Example 1 (continued) *For the case in which match values are distributed uniformly on* $[\lambda, 1]$*, the distribution function is* $F(\varepsilon,\lambda) = (\varepsilon-\lambda)/(1-\lambda)$ *and Eq. (22) can be solved for* ε *to obtain:*

$$\varepsilon = 1 - \frac{4\lambda}{(1-\lambda)^2} \tag{24}$$

The RHS of Eq. (24) is decreasing in λ *and a direct comparison with the RHS of Eq. (11) immediately shows that the incentives to invest are excessive from the point of view of social welfare maximisation. Because of this, consumers search too much.*
We illustrate this in Figure 6, which adds the decreasing schedule $\hat{\varepsilon}_3(\lambda)$ *to Figure 8(a). The new decreasing schedule depicts the RHS of (24). Recall that the intersection between* $\hat{\varepsilon}_1(\lambda)$ *and* $\hat{\varepsilon}_2(\lambda)$ *gives the private equilibrium* $\{\hat{\varepsilon}^*, \lambda^*\}$*. The crossing point between* $\hat{\varepsilon}_1(\lambda)$ *and* $\hat{\varepsilon}_3(\lambda)$ *gives the social optimum* $\{\hat{\varepsilon}^o, \lambda^o\}$*. There is too much search and too much investment.*

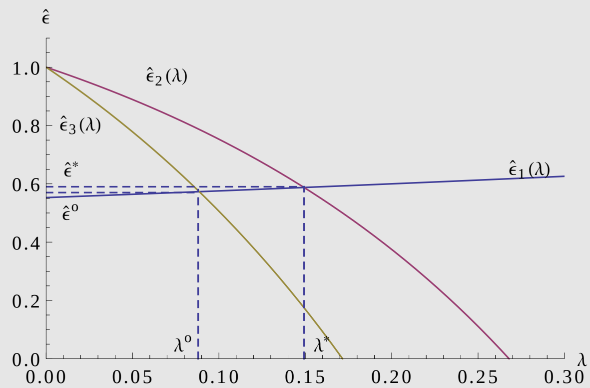


Figure 6: Market equilibrium and social optimum: uniform distribution on $[\lambda, 1]$.

Another example for which we obtain over-investment is when match values are distributed according to the Kumaraswamy (1980) distribution $F(\varepsilon,\lambda) = 1 - \left[1 - \frac{\varepsilon-\lambda}{1-\lambda}\right]^{\alpha}$, $\varepsilon \in [\lambda, 1]$, $\alpha \leq 1$. As shown above, this distribution is supermodular and has an increasing density.

We now provide an alternative characterisation of over-investment. For this, we invoke the notions of mean-residual-life of a random variable and the mean-residual-life stochastic ordering.

Definition 2 *The mean-residual-life (MRL) function of the random variable ε is defined as:*

$$MRL(\hat{\varepsilon},\lambda) = E[\varepsilon - \hat{\varepsilon} | \varepsilon > \hat{\varepsilon}] = \frac{\int_{\hat{\varepsilon}} \varepsilon f(\varepsilon,\lambda) d\varepsilon}{1 - F(\hat{\varepsilon},\lambda)} - \hat{\varepsilon}.$$

The MRL function is used much in industrial engineering: in our model, it basically gives the expectation of the difference between the valuation of a consumer who stops searching and the reservation value.

Proposition 5 *If the distribution of match values $F(\varepsilon,\lambda)$ has a MRL that is weakly decreasing in λ, and $1 - F(\varepsilon,\lambda)$ is log-concave in ε, then the market over-provides quality.*

Proof. See the Appendix. ■

The intuition behind this result is as follows. The notion of MRL refers to the expectation of the match value conditional on stopping. When the MRL is decreasing in quality investment, this means that a higher quality reduces this conditional expectation. Although more consumers stop search and buy, their average value goes down in quality investment. For failure functions that are log-concave, the frequency of consumers at the margin is not very large. The firms, whose incentives are governed by he raise in the stopping probability per marginal consumer, invests too much.

Before moving into an example to illustrate Proposition 4, we observe that for FOSD to be compatible with a MRL decreasing in λ, the lower bound of the support must be strictly increasing in λ. To see this, notice first that FOSD requires the lower bound to be weakly increasing in λ. By contradiction, suppose the lower bound of the support of F remains constant when λ increases. Because the MRL is decreasing in λ, if $\lambda_1 < \lambda_2$, then for any ε it holds that:

$$\frac{\int_{\varepsilon}(1 - F(z,\lambda_1)dz)}{1 - F(\varepsilon,\lambda_1)} > \frac{\int_{\varepsilon}(1 - F(z,\lambda_2))dz}{1 - F(\varepsilon,\lambda_2)}$$

This inequality holds for any ε, in particular for the lower bound $\underline{\varepsilon}$, which is independent of λ. For $\varepsilon = \underline{\varepsilon}$, we have $F(\underline{\varepsilon},\lambda_1) = F(\underline{\varepsilon},\lambda_2) = 0$. Then, the above inequality implies that:

$$\int_{\underline{\varepsilon}}(1 - F(z,\lambda_1))dz > \int_{\underline{\varepsilon}}(1 - F(z,\lambda_2))dz.$$

This, however, contradicts FOSD because $1-F(\varepsilon,\lambda_1) \leq 1-F(\varepsilon,\lambda_2)$ for any ε. So, we conclude that the lower bound of F must increase in λ.

Example 5 ***Decreasing mean-residual-life***
Consider the case in which match values follow the distribution:

$$F(\varepsilon,\lambda) = \frac{(\varepsilon-\lambda)(1-\lambda\varepsilon)}{(1-\lambda)^2}, \text{ with support } [\lambda,1],\ 0 \leq \lambda < 1.$$

The case of $\lambda = 0$ is the case of the uniform distribution on $[0,1]$. Taking the derivative with respect to λ gives $\frac{\partial[1-F(\varepsilon,\lambda)]}{\partial\lambda} = \frac{(1-\varepsilon)^2(1-\lambda^2)}{(1-\lambda)^3} \geq 0$, so match values increase in the sense of FOSD when λ goes up.
The corresponding density function is:

$$f(\varepsilon,\lambda) = \frac{1-2\lambda\varepsilon+\lambda^2}{(1-\lambda)^2}.$$

Notice that this density is decreasing in ε so this example violates the conditions in Proposition 4. Yet, this example satisfies the conditions in Proposition 5 and therefore the market over-invests in quality from a social welfare perspective.
To see this, notice first that the density $f(\varepsilon,\lambda)$ is log-concave in ε. In fact,

$$\frac{\partial}{\partial\varepsilon}\frac{\partial Log[f(\varepsilon,\lambda)]}{\partial\varepsilon} = -\frac{4\lambda^2}{(1+\lambda^2-2\lambda\epsilon)^2} < 0.$$

It is known that the log-concavity of f implies the log-concavity of $1-F$.
We now compute the MRL of ε:

$$MRL(\varepsilon,\lambda) = \frac{\int_\varepsilon^1 1-\frac{(z-\lambda)(1-\lambda z)}{(1-\lambda)^2}dz}{1-\frac{(\varepsilon-\lambda)(1-\lambda\varepsilon)}{(1-\lambda)^2}} = \frac{(1-\varepsilon)(3-\lambda(4+2\varepsilon-3\lambda))}{6[1-\lambda(1+\varepsilon-\lambda)]}.$$

Taking the derivative with respect to λ gives:

$$\frac{\partial MRL}{\lambda} = -\frac{\left(1-\lambda^2\right)(1-\varepsilon)^2}{6\left(1-\lambda(1+\varepsilon)+\lambda^2\right)^2} < 0,$$

so the MRL is strictly decreasing in λ.

The next example also fails to satisfy the conditions in Proposition 4. However, by virtue of Proposition 5 we can establish that the market provides an excessive amount of quality and correspondingly consumers search too much.

Example 6 ***Kumaraswamy distribution*** $F(\varepsilon,\lambda) = 1-\left[1-\frac{\varepsilon-\lambda}{1-\lambda}\right]^\alpha,\ \varepsilon \in [\lambda,1],\ \alpha > 1.$

The density function is

$$f(\varepsilon,\lambda) = \frac{\alpha\left(1-\frac{\varepsilon-\lambda}{1-\lambda}\right)^{\alpha-1}}{1-\lambda}, \tag{25}$$

When $\alpha > 1$, this density function is decreasing in ε so we cannot apply Proposition 4. However, note that

$$\frac{\partial}{\partial\varepsilon}\frac{\partial Log[f(\varepsilon,\lambda)]}{\partial\varepsilon} = -\frac{b}{(1-x)^2} < 0,$$

which implies that $1-F$ is log-concave in ε. For this distribution, the MRL of ε is:

$$MRL(\varepsilon,\lambda) = \frac{\int_\varepsilon^1\left(1-\frac{z-\lambda}{1-\lambda}\right)^b dz}{\left(1-\frac{\varepsilon-\lambda}{1-\lambda}\right)^b} = \frac{1-\varepsilon}{b+1},$$

which is weakly decreasing in λ.

We conclude that the conditions of Proposition 5 apply and therefore the market invests too much from a social welfare perspective. □

6.2 A sufficient condition for under-investment

In this subsection, we give a sufficient condition for under-investment. The condition is the counterpart of that in Proposition 4, and is therefore intuitive and easy to verify.

Proposition 6 *If the distribution of match values $F(\varepsilon,\lambda)$ is submodular and the density weakly decreases in ε, then the market under-supplies quality.*

Proof. The proof is similar to the proof of Proposition 4 and therefore omitted. ■

We now return to the previous examples and check that firms invest too little from the perspective of social welfare maximization. As a result, consumers search too little.

Example 3 (continued) *For the case in which match values are distributed uniformly on $[0,1+\lambda]$, it is easy to verify that the conditions in Proposition 6 hold. In fact, the density is constant in ε and strictly decreasing in λ. As a result, the market under-provides quality. Eq. (22) becomes:*

$$\frac{\lambda+\varepsilon+1}{2\lambda+2} - \lambda = 0,$$

which can be solved for ε to obtain

$$\varepsilon = \lambda(1+2\lambda) - 1.$$

This function is clearly increasing in λ. Moreover, it is clearly below the equation characterising the market equilibrium (15) for the relevant range $\lambda < \sqrt{2/3}$. This implies that *the market under-provides quality. Figure 7 illustrates. The crossing point between $\hat{\varepsilon}_1(\lambda)$ and $\hat{\varepsilon}_2(\lambda)$ gives the market equilibrium; the crossing point between $\hat{\varepsilon}_1(\lambda)$ and $\hat{\varepsilon}_3(\lambda)$ gives the optimum. We see that $\lambda^o > \lambda^*$ so there is under-investment; as a consequence, $\hat{\varepsilon}^o > \hat{\varepsilon}^*$, which means that consumers search too little in equilibrium.*

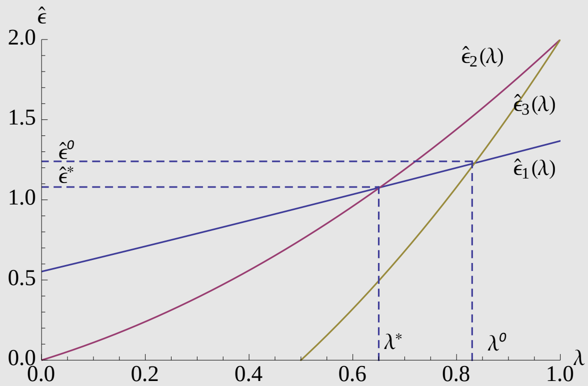


Figure 7: Market equilibrium and social optimum; $F(\varepsilon) = \frac{\varepsilon}{1+\lambda}$.

7 Discussion

7.1 The additive case

In this section, we discuss the special case in which quality enters the match value distribution additively. This case has been studied by Wolinsky (2005) and by Fishman and Levy (2015), although in different contexts. The additive case is special because changes in λ do not affect the shape of the density, but only its location.

Proposition 7 *If the distribution of match values $F(\varepsilon, \lambda) = F(\varepsilon - \lambda)$, then the market provision of quality is optimal.*

Proof. See the Appendix. ■

The additive case moves the distribution of match values in a special way. As a matter of fact, when quality enters the distribution of match values additively, the reservation value of consumers is affected exactly in the same additive way; that is, if the pre-investment reservation value is $\hat{\varepsilon}$, the post-investment is $\hat{\varepsilon} + \lambda$ (details in the Appendix). This means that neither demand nor the equilibrium price vary with the investment. In this case, the way the private gains and the social gains are affected by the investment coincide, and consequently private investment is optimal.

7.2 Monopoly

We conclude this section by studying the case in which a monopolist serves the whole market. Suppose that a single seller controls the prices of all the products in the market, while consumers search as usual, incurring a search cost c each time they inspect one product.

Proposition 8 *A monopolist supplies the socially optimal amount of quality.*

Proof. See the Appendix. ■

The key to an understanding of this result is that a monopolist does not face competition other than from its own products. As a result, it has an incentive to continue to increase its price till it equals the reservation price of consumers, therefore $p^* = \hat{\varepsilon}$. Therefore, in deciding its investment in quality, the monopolist looks at how the reservation value changes in quality. Using the search rule:

$$\int_{\hat{\varepsilon}} (1 - F(z, \lambda))dz - c = 0,$$

it turns out that

$$\frac{\partial \hat{\varepsilon}}{\partial \lambda} = \frac{\int_{\hat{\varepsilon}} \frac{\partial (1-F(z,\lambda)}{\partial \lambda} dz}{1 - F(\hat{\varepsilon}, \lambda)},$$

which is exactly what the social planner takes into account.

8 Conclusion

This paper has studied the market provision of quality in a consumer search market for differentiated products. The analysis has clarified how the incentives to invest in quality depend on search costs and how they relate to the social incentives. Depending on the nature of the distribution of match values, higher search costs can result in higher or in lower quality. In particular, when the distribution of match values is supermodular and has an increasing density, then firms increase their quality investments as search costs go up. It is precisely in such settings where quality investments are excessive from a social welfare point of view. By contrast, when the distribution of match values is submodular and has a decreasing density firms lower their quality as search costs increase and the amount of quality supplied in the market is insufficient.

Our results are relevant to understand how innovations that reduce search costs such as the Internet may affect the incentives to provide quality in markets. For example, digital platforms such as the online travel agents reduce considerably the costs consumers incur to search for satisfactory products. Whether firms respond to such developments by improving their products or not depends on the relationship between the distribution of consumer valuations and quality.

Appendix

Proof of Lemma 1.

Consider the function of λ^* defined by the RHS of (7). At $\lambda^* = 0$, this function is strictly positive because $K'(0) = 0$ and $\frac{\partial(1-F(\hat{\varepsilon},0))}{\partial\lambda} > 0$. For $K(\cdot)$ sufficiently convex, the RHS of (7) will eventually turn decreasing and cross a single time the horizontal axis. Once we obtain the unique solution of (7) in λ^*, we can plug it into (6) to obtain the unique p^*.

The candidate firm equilibrium (p^*, λ^*) that solves (6) and (7) is indeed an equilibrium because with $K(\lambda)$ sufficiently convex and with f non-decreasing, the payoff function (3) is jointly concave in both p_i and λ_i. To see this, we calculate the Hessian matrix:

$$H = \begin{bmatrix} -\frac{2f(\hat{\varepsilon}-p^*+p_i,\lambda_i)+p_i f'(\hat{\varepsilon}-p^*+p_i,\lambda_i)}{1-F(\hat{\varepsilon},\lambda^*)} & \frac{\frac{\partial(1-F(\hat{\varepsilon}-p^*+p_i,\lambda_i))}{\partial\lambda}-p_i\frac{\partial f(\hat{\varepsilon}-p^*+p_i,\lambda_i)}{\partial\lambda}}{1-F(\hat{\varepsilon},\lambda^*)} \\ \frac{\frac{\partial(1-F(\hat{\varepsilon}-p^*+p_i,\lambda_i))}{\partial\lambda}-p_i\frac{\partial f(\hat{\varepsilon}-p^*+p_i,\lambda_i)}{\partial\lambda}}{1-F(\hat{\varepsilon},\lambda^*)} & \frac{p_i\frac{\partial^2(1-F(\hat{\varepsilon}-p^*+p_i))}{\partial\lambda^2}}{1-F(\hat{\varepsilon},\lambda^*)} - K''(\lambda_i) \end{bmatrix} \tag{26}$$

With f non-decreasing in ε, the first leading principal minor of the matrix H is clearly negative. The second leading principal minor of the matrix H can be made positive with $K(\cdot)$ sufficiently convex so that the payoff function is strictly concave in both p_i and λ_i. ■

Proof of Proposition 1. Part (a). By the implicit function theorem, Eq. (8) defines a relation $\hat{\varepsilon}_1(\lambda^*)$. At $\lambda^* = 0$, $\hat{\varepsilon}_1(0)$ is strictly positive because $F(\varepsilon, 0)$ is a well-defined distribution. Moreover, $\hat{\varepsilon}_1(\lambda^*)$ is increasing in λ^*. This, which follows from the idea that there is more search when the products have higher quality, can be seen by first rewriting Eq. (8) by using integration by parts as follows:

$$h(\hat{\varepsilon}, \lambda^*) \equiv \int_{\hat{\varepsilon}} (1 - F(z, \lambda^*))dz - c = 0.$$

Then, using the implicit function theorem gives:

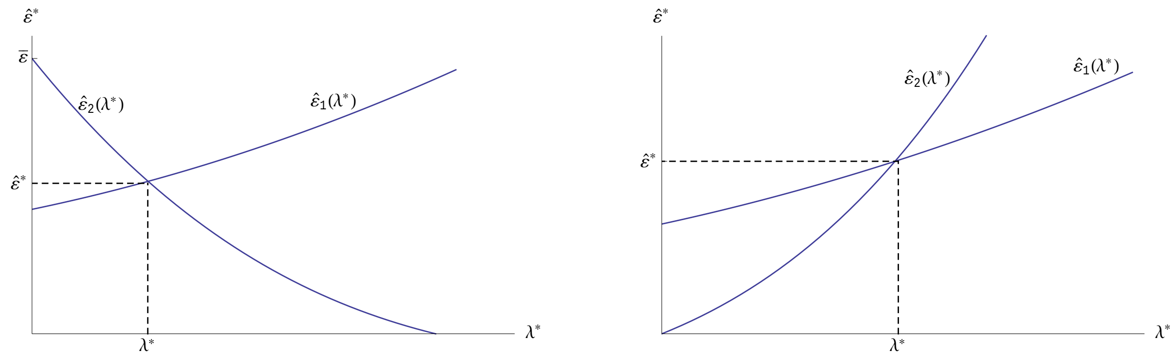
$$\frac{\partial\hat{\varepsilon}_1}{\partial\lambda^*} = -\frac{\frac{\partial h}{\partial\lambda^*}}{\frac{\partial h}{\partial\hat{\varepsilon}}} = \frac{\int_{\hat{\varepsilon}} \frac{\partial(1-F(z,\lambda^*))}{\partial\lambda^*}dz}{1 - F(\hat{\varepsilon}, \lambda^*)} > 0.$$

Secondly, we note that, by Lemma 1, when $K(\lambda)$ is sufficiently convex, a solution to Eq. (9) exists. This solution defines a relation $\hat{\varepsilon}_2(\lambda^*)$, which, under assumption (a), decreases in λ^*. To see this, we apply the implicit function theorem to Eq. (9):

$$\frac{\partial\hat{\varepsilon}_2}{\partial\lambda^*} = -\frac{\frac{1}{f^2}\left(\frac{\partial^2(1-F)}{\partial\lambda^{*2}}f - \frac{\partial(1-F)}{\partial\lambda^*}\frac{\partial f}{\partial\lambda^*}\right) - K''(\lambda^*)}{\frac{1}{f^2}\left(-\frac{\partial f}{\partial\lambda^*}f - \frac{\partial(1-F)}{\partial\lambda^*}\frac{\partial f}{\partial\hat{\varepsilon}}\right)}.$$

Under assumption (a), the denominator of $\partial\hat{\varepsilon}_2/\partial\lambda^*$ is negative. If K is sufficiently convex, the numerator is negative, thus the implicity function $\hat{\varepsilon}_2(\lambda^*)$ decreases in λ^*. Moreover, from the assumption, it follows that when $\lambda^* = 0$, $\hat{\varepsilon}_2(0) = \bar{\varepsilon} \geq \hat{\varepsilon}_1(0) > 0$, as mentioned above. As a result, $\hat{\varepsilon}_1(\lambda^*)$ and $\hat{\varepsilon}_2(\lambda^*)$ cross once and only once, guaranteeing a unique solution. Figure 8(a) illustrates this case.

Part (b). Under assumption (b), the denominator of $\partial\hat{\varepsilon}_2/\partial\lambda^*$ is positive and by sufficient convexity of K the numerator is negative. Therefore, $\hat{\varepsilon}_2(\lambda^*)$ increases in λ^*. Further, at $\lambda^* = 0$, $\hat{\varepsilon}_2(0)$ is equal to zero. Furthermore, under sufficient convexity of K, the assumptions ensure that $\hat{\varepsilon}_2(\lambda)$ is sufficiently increasing, which implies that $\frac{\partial\varepsilon_1^*}{\partial\lambda^*} < \frac{\partial\varepsilon_2^*}{\partial\lambda^*}$. Therefore $\hat{\varepsilon}_1(\lambda^*)$ and $\hat{\varepsilon}_2(\lambda^*)$ surely cross one another only once, which guarantees the existence of a unique solution. See Figure 8(b) illustrates this case.



(a) $\frac{\partial\hat{\varepsilon}_2}{\partial\lambda^*} < 0$ (b) $\frac{\partial\hat{\varepsilon}_2}{\partial\lambda^*} > 0$

Figure 8: Existence and uniqueness of equilibrium

■

Proof of Proposition 2.

Note that the search cost only enters the market equilibrium through the search rule given by Eq. (8), and has no bearing on Eq. (9). Therefore, only the function $\hat{\varepsilon}_1(\lambda^*)$, defined implicitly by Eq. (8), is affected by changes in search costs. Holding λ^* unchanged, an increase in c must be compensated by a decrease in $\hat{\varepsilon}$. Mathematically,

$$\frac{d\hat{\varepsilon}}{dc} = -\frac{\frac{\partial h}{\partial c}}{\frac{\partial h}{\partial \hat{\varepsilon}}} = -\frac{1}{1 - F(\varepsilon, \lambda^*)} < 0.$$

Hence, after an increase in search cost, the function $\hat{\varepsilon}_1(\lambda^*)$ will shift downwards. This, given the assumed monotonicity of $\hat{\varepsilon}_2(\lambda^*)$ will result in a new market equilibrium in which $\hat{\varepsilon}$ decreases.

(a) Under the conditions in Proposition 1(a), $\hat{\varepsilon}_2(\lambda^*)$ is decreasing. It follows, thus, that $\hat{\varepsilon}$ decreases in c while λ^* increases in c. Because the density f is non-decreasing in ε, a decrease $\hat{\varepsilon}$ tends to lower the price. Still, the behavior of the equilibrium price is ambiguous because the sign of $\partial p^*/\partial\lambda^*$ can be positive or negative. In fact, in this case we have:

$$\frac{dp^*}{dc} = \underbrace{\frac{\partial p^*}{\partial\hat{\varepsilon}}}_{\leq 0}\underbrace{\frac{\partial\hat{\varepsilon}}{\partial c}}_{<0} + \underbrace{\frac{\partial p^*}{\partial\lambda^*}}_{?}\underbrace{\frac{\partial\lambda^*}{\partial c}}_{>0}. \tag{27}$$

For distributions satisfying the hazard-ratio ordering, the hazard rate $f/(1-F)$ decreases in λ^* and it then follows straightforwardly that $\partial p^*/\partial\lambda^* \geq 0$ and therefore the sign of Eq. (27) is unambiguously positive.

(b) Under the conditions in Proposition 1(b), the function $\hat{\varepsilon}_2(\lambda^*)$ increases and we therefore conclude that both $\hat{\varepsilon}$ and λ^* decrease in c. The behaviour of the equilibrium price is now given by the derivative:

$$\frac{dp^*}{dc} = \underbrace{\frac{\partial p^*}{\partial\hat{\varepsilon}}}_{\leq 0}\underbrace{\frac{\partial\hat{\varepsilon}}{\partial c}}_{<0} + \underbrace{\frac{\partial p^*}{\partial\lambda^*}}_{?}\underbrace{\frac{\partial\lambda^*}{\partial c}}_{<0}, \tag{28}$$

where the difference with Eq. (27) is that the last term is negative now. Because the first summand of the RHS of (28) is positive an the second is ambiguous, we cannot conclude anything about the effect of higher search costs on the price. For the exponential distribution $F = 1-e^{-\frac{\varepsilon}{1+\lambda}}$, defined on $[0,\infty]$, it holds that $\partial p^*/\partial\hat{\varepsilon} = 0$ because the hazard rate is constant in ε. Moreover, because this distribution satisfies the hazard rate ordering, we have $\partial p^*/\partial\lambda^* \geq 0$. We then conclude that the equilibrium price decreases in search cost. ■

Proof of Proposition 3.

Because, consumers search efficiently for a fixed λ, to prove the result we only need to compare Eqs. (9) and (22). Taking the difference of the two first-order conditions gives:

$$\frac{\partial W}{\partial\lambda} - \frac{\partial\pi}{\partial\lambda} = \frac{\int_{\hat{\varepsilon}}\frac{\partial(1-F(z,\lambda))}{\partial\lambda}dz}{1-F(\hat{\varepsilon},\lambda)} - \frac{\frac{\partial(1-F(\hat{\varepsilon},\lambda))}{\partial\lambda}}{f(\hat{\varepsilon},\lambda)}. \tag{29}$$

When (29) is positive, the solution to (22), which we denoted as $\hat{\varepsilon}_3(\lambda)$, is smaller than the solution to (9), which we have denoted up until now by $\hat{\varepsilon}_2(\lambda)$. As a result, the crossing point between $\hat{\varepsilon}_1(\lambda)$ and $\hat{\varepsilon}_2(\lambda)$ will be below the crossing point between $\hat{\varepsilon}_1(\lambda)$ and $\hat{\varepsilon}_3(\lambda)$. This implies under-investment. When (29) is negative, we have over-investment. ■

Proof of Proposition 4.

Because F is supermodular, the density $f(\varepsilon,\lambda)$ increases in λ. This implies that $\frac{\partial(1-F(\varepsilon,\lambda))}{\partial\lambda}$ decreases in ε. Therefore, for any ε and $z > \varepsilon$, we have:

$$\frac{\partial(1-F(\varepsilon,\lambda))}{\partial\lambda} > \frac{\partial(1-F(z,\lambda))}{\partial\lambda}$$

Taking the right hand integral in the inequality above gives:

$$\int_{\varepsilon}^{\overline{\varepsilon}} \frac{\partial(1-F(\varepsilon,\lambda))}{\partial\lambda} dz > \int_{\varepsilon}^{\overline{\varepsilon}} \frac{\partial(1-F(z,\lambda))}{\partial\lambda} dz,$$

or

$$\frac{\partial(1-F(\varepsilon,\lambda))}{\partial\lambda} \int_{\varepsilon}^{\overline{\varepsilon}} dz > \int_{\varepsilon}^{\overline{\varepsilon}} \frac{\partial(1-F(z,\lambda))}{\partial\lambda} dz. \tag{30}$$

Similarly, because $f(\varepsilon,\lambda)$ weakly increases in ε, for all $z > \varepsilon$, we have:

$$f(z,\lambda) \geq f(\varepsilon,\lambda).$$

Taking the right hand integral gives,

$$\int_{\varepsilon}^{\overline{\varepsilon}} f(z,\lambda) dz \geq \int_{\varepsilon}^{\overline{\varepsilon}} f(\varepsilon,\lambda) dz,$$

or

$$1 - F(\varepsilon,\lambda) \geq f(\varepsilon,\lambda) \int_{\varepsilon}^{\overline{\varepsilon}} dz. \tag{31}$$

By combining (30) and (31), we get:

$$\frac{\frac{\partial(1-F(\varepsilon,\lambda))}{\partial\lambda} \int_{\varepsilon}^{\overline{\varepsilon}} dz}{f(\varepsilon,\lambda)\int_{\varepsilon}^{\overline{\varepsilon}} dz} = \frac{\frac{\partial(1-F(\varepsilon,\lambda))}{\partial\lambda}}{f(\varepsilon,\lambda)} > \frac{\int_{\varepsilon}^{\overline{\varepsilon}} \frac{\partial(1-F(z,\lambda))}{\partial\lambda} dz}{1-F(\varepsilon,\lambda)},$$

which is the same as the over-investment condition in Proposition 3. ■

Proof of Proposition 5.

After integration by parts, we can rewrite the MRL function more conveniently as follows:

$$MRL(\hat{\varepsilon};\lambda) = \frac{\int_{\hat{\varepsilon}} (1-F(z;\lambda)) dz}{1-F(\hat{\varepsilon};\lambda)}.$$

Because the mean residual life of $F(\varepsilon,\lambda)$ is weakly decreasing in λ, its first-order derivative with respect to λ is non-positive:

$$\frac{[1-F(\varepsilon,\lambda)] \int_{\varepsilon} \frac{\partial(1-F(z,\lambda))}{\partial\lambda} dz - \frac{\partial(1-F(\varepsilon,\lambda))}{\partial\lambda} \int_{\varepsilon} [1-F(z,\lambda)] dz}{[1-F(\varepsilon,\lambda)]^2} \leq 0.$$

Rearranging this expression, we get:

$$\frac{\int_{\varepsilon} \frac{\partial(1-F(z,\lambda))}{\partial\lambda} dz}{1-F(\varepsilon,\lambda)} \leq \frac{\frac{\partial(1-F(\varepsilon,\lambda))}{\partial\lambda}}{\frac{(1-F(\varepsilon,\lambda))^2}{\int_{\varepsilon}(1-F(z,\lambda))dz}}. \tag{32}$$

Now, because $1 - F(\lambda,\varepsilon)$ is log-concave in ε, the right hand integral of the function $\int_{\varepsilon}(1 - F(z,\lambda))dz$ is also log-concave in ε (see Theorem 3, Bagnoli and Bergstrom, 2005). Therefore the second-order derivative of $\log(\int_{\varepsilon}(1 - F(z,\lambda))dz)$ with respect to ε is negative:

$$\frac{f(\varepsilon,\lambda)\int_{\varepsilon}(1-F(z,\lambda))dz - (1-F(\varepsilon,\lambda))^2}{[\int_{\varepsilon}(1-F(z,\lambda))dz]^2} < 0.$$

Because the denominator of this expression is positive, this implies that:

$$f(\varepsilon, \lambda) < \frac{(1 - F(z, \lambda))^2}{\int_\varepsilon (1 - F(z, \lambda)) dz} \tag{33}$$

Combining (32) and (33) gives:

$$\frac{\int_\varepsilon \frac{\partial(1-F(z,\lambda))}{\partial \lambda} dz}{1 - F(\varepsilon, \lambda)} \leq \frac{\frac{\partial(1-F(\varepsilon,\lambda))}{\partial \lambda}}{\frac{(1-F(\varepsilon,\lambda))^2}{\int_\varepsilon (1-F(z,\lambda))dz}} < \frac{\frac{\partial(1-F(\varepsilon,\lambda))}{\partial \lambda}}{f(\varepsilon, \lambda)},$$

which, by Proposition 3, implies over-investment. ■

Proof of Proposition 7.

Let $\hat{\varepsilon}_0^*$ be the consumers reservation value when $\lambda = 0$, that is, the solution to:

$$\int_{\hat{\varepsilon}_0} (z - \varepsilon_0) dF(z) = c. \tag{34}$$

Likewise, let ε_λ^* the consumers reservation value for $\lambda > 0$, that is the solution to

$$\int_{\varepsilon_\lambda} (z - \varepsilon_\lambda) dF(z - \lambda) = c$$

By the change of variables $t = z - \lambda$, this is equivalent to:

$$\int_{\varepsilon_\lambda - \lambda} (t - (\varepsilon_\lambda - \lambda)) dF(t) = c. \tag{35}$$

From Eqs. (34) and (35), it readily follows that $\hat{\varepsilon}_\lambda^* = \hat{\varepsilon}_0^* + \lambda$. Therefore, in this case in which quality enters the distribution of match values additively, the reservation value is affected also additively by the investment in quality.

The firms' equilibrium investment is given by the condition (7):

$$\frac{\frac{\partial(1-F(\hat{\varepsilon}_\lambda^*,\lambda))}{\partial \lambda}}{f(\hat{\varepsilon}_\lambda^*, \lambda)} = K'(\lambda)$$

Because $F(\hat{\varepsilon}_\lambda^*, \lambda) = F(\hat{\varepsilon}_\lambda^* - \lambda) = F(\hat{\varepsilon}_0^*)$ and similarly $f(\hat{\varepsilon}_\lambda^* - \lambda) = f(\hat{\varepsilon}_0^*)$, we have:

$$\frac{\partial(1 - F(\hat{\varepsilon}_\lambda^*, \lambda))}{\partial \lambda} = f(\hat{\varepsilon}_\lambda^* - \lambda).$$

Using this relation, we can rewrite the equilibrium condition (7) simply as:

$$1 = K'(\lambda)$$

The planner's investment problem is given by the solution to the FOC:

$$\frac{\int_{\hat{\varepsilon}} \frac{\partial(1-F(z-\lambda))}{\partial \lambda} dz}{1 - F(\hat{\varepsilon} - \lambda)} = K'(\lambda)$$

Using the relations above, this can be rewritten as:

$$1 = K'(\lambda),$$

which shows that the firms and the planner have the same incentives. ■

Proof of Proposition 8.

Because the search rule does not depend on the price at which the products sell, for a fixed investment, the reservation value is computed as before, that is, as the solution to the equation:

$$\int_{\hat{\varepsilon}} (z - \hat{\varepsilon}) f(z, \lambda) - c = 0.$$

Suppose the monopolist charges a price p for each of the products. The profits of the monopolist are then equal to

$$\pi = p - K(\lambda) \tag{36}$$

provided that $\hat{\varepsilon} \geq p$ for otherwise consumers would not participate in the market. This implies that the monopolist will continue to increase its price until $p = \hat{\varepsilon}$.

The first order condition for profits maximization with respect to λ is then:

$$\frac{\partial \hat{\varepsilon}}{\partial \lambda} - K'(\lambda) = 0. \tag{37}$$

Using the search rule, we can derive that (for details see the proof of Proposition 1):

$$\frac{\partial \hat{\varepsilon}}{\partial \lambda} = \frac{\int_{\hat{\varepsilon}} \frac{\partial (1 - F(z, \lambda))}{\partial \lambda} dz}{1 - F(\hat{\varepsilon}, \lambda)} \tag{38}$$

so the FOC (37) becomes

$$\frac{\int_{\hat{\varepsilon}} \frac{\partial (1 - F(z, \lambda))}{\partial \lambda} dz}{1 - F(\hat{\varepsilon}, \lambda)} - K'(\lambda) = 0.$$

This expression is exactly identical to the FOC for the planner in Eq. (22). We then conclude that the monopolist provides a socially optimal amount of quality. ■

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